Intro to Modern Algebra I
Math GU4041
New York, 2021/02/03

EXERCISE SHEET 4

Permutations

Exercise 1. List all the elements of $\mathfrak{S}_2$ and $\mathfrak{S}_3$. Verify that all their non-identity elements are cycles.

Exercise 2. Prove the following statements.

(a) Two disjoint cycles commute: $\sigma\tau = \tau\sigma$.

(b) If σ is a k-cycle, then for some $i \in \{1, \ldots, n\}$,

$$\sigma = (i \;\; \sigma(i) \;\; \sigma^2(i) \;\; \ldots \;\; \sigma^{k-1}(i))\,.$$

(c) If σ is a k-cycle, then $o(\sigma) = k$.

Exercise 3. Given a fixed $\sigma \in \mathfrak{S}_n$, show that the relation defined by

$$i \sim j \Leftrightarrow \exists k \in \mathbb{Z} \text{ s.t. } \sigma^k(i) = j\,.$$

is an equivalence relation. Conclude that the orbits relative to σ form a partition of $\{1, \ldots, n\}$.

Exercise 4. For every one of the following permutations, find the cycle decomposition, compute the order, and express the permutation as a product of transpositions.

(a) $(1 \;\; 4 \;\; 2)(1 \;\; 4 \;\; 3)$.

(b) $(1 \;\; 2)(1 \;\; 3)$.

(c) $(1 \;\; 2 \;\; 3)(1 \;\; 3 \;\; 2)$.

(d) $(1 \;\; 2 \;\; 3 \;\; 5 \;\; 7)(2 \;\; 4 \;\; 7 \;\; 6)$.

(e) $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 3 & 1 & 4 & 2 & 7 & 6 & 9 & 8 & 5 \end{pmatrix}$

Exercise 5. Write permutations in $\mathfrak{S}_{13}$ of order 13, 42, 20. What is the permutation in $\mathfrak{S}_{13}$ of the greatest order?

Exercise 6. Compute the following:

$$\min \{ \ m \in \mathbb{N} \setminus \{0\} \ \mid \ \forall \sigma \in \mathfrak{S}_4, \ \sigma^m = \mathrm{Id} \ \}$$

Exercise 7. Complete the proof of the lemma stating that if $\sigma \in \mathfrak{S}_n$ and $\tau = (i \quad j)$ is a transposition, the number of orbits in σ and $\tau\sigma$ differ by 1. In the case when i, j are in different orbits, cover the cases when i or j or both are alone in their orbit. In the case when i, j are in the same orbit, cover the cases when they are adjacent in the cycle, or when their orbit has just two elements.

Exercise 8. Prove that every non-identity element of the alternating group $\mathfrak{A}_n$ is a product of 3-cycles. (Hint: Use Exercise 4, (a),(b),(c) to show that, if $n \geq 3$, a product of two transpositions is either a 3-cycle or a product of two 3-cycles.)