COLUMBIA UNIVERSITY
IN THE CITY OF NEW YORK

Intro to Modern Algebra I
Math GU4041
New York, 2021/02/10

EXERCISE SHEET 5

Isometries

Exercise 1. Find matrices $A, B, C \in GL(2, \mathbb{R})$, where A, B are not diagonal and C is diagonal such that

$$AB = BA\,,$$
$$AC \neq CA\,.$$

Exercise 2. Define

$$R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\,.$$

Note that $R_\theta = R_{\theta+2\pi n}$, with $n \in \mathbb{Z}$. Consider the subset

$$SO(2) = \{\ R_\theta \ \mid\ \theta \in \mathbb{R}\ \} \subset SL(2, \mathbb{R})\,.$$

Prove that $SO(2) < SL(2, \mathbb{R})$ (i.e. that it is a subgroup). In order to do so, compute $R_\theta R_\phi$ and R_θ^{-1}.

Exercise 3. Define

$$S_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}\,.$$

Note that $S_\theta = S_{\theta+2\pi n}$, with $n \in \mathbb{Z}$. Consider the subset

$$O(2) = SO(2) \cup \{\ S_\theta \ \mid\ \theta \in \mathbb{R}\ \} \subset GL(2, \mathbb{R})\,.$$

Prove that $O(2) < GL(2, \mathbb{R})$ (i.e. that it is a subgroup). In order to do so, compute $S_\theta R_\phi$, $R_\theta S_\phi$, $S_\theta S_\phi$ and S_θ^{-1}. Notice that $O(2)$ is not a subgroup of $SL(2, \mathbb{R})$.

Exercise 4. Prove that

$$SO(2) = \{\ A \in SL(2, \mathbb{R})\ \mid\ AA^T = \mathrm{Id}\ \}\,,$$
$$O(2) = \{\ A \in GL(2, \mathbb{R})\ \mid\ AA^T = \mathrm{Id}\ \}\,.$$

$O(2)$ is called the **orthogonal group** of $\mathbb{R}^2$, and $SO(2)$ the **special orthogonal group** of $\mathbb{R}^2$.

Exercise 5. Show that S_θ is a reflection, and compute the axis of the reflection. (Hint: for example, you can compute the eigenvalues and the eigenvectors of S_θ).

Exercise 6. Consider the subset

$$\mathrm{Isom}(\mathbb{R}^2) = \left\{ \begin{pmatrix} \multicolumn{2}{c}{\multirow{2}{*}{A}} & v_x \\ & & v_y \\ 0 & 0 & 1 \end{pmatrix} \;\middle|\; A \in O(2), \quad v_x, v_y \in \mathbb{R} \right\} \subset GL(3,\mathbb{R})\,.$$

Prove that $\mathrm{Isom}(\mathbb{R}^2) < GL(3,\mathbb{R})$. In order to do so, verify that

$$\begin{pmatrix} \multicolumn{2}{c}{A} & v_x \\ & & v_y \\ 0 & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} \multicolumn{2}{c}{A^{-1}} & w_x \\ & & w_y \\ 0 & 0 & 1 \end{pmatrix},$$

where

$$\begin{pmatrix} w_x \\ w_y \end{pmatrix} = -A^{-1} \begin{pmatrix} v_x \\ v_y \end{pmatrix}.$$

Exercise 7. Prove that the subset

$$\left\{ \begin{pmatrix} 1 & 0 & v_x \\ 0 & 1 & v_y \\ 0 & 0 & 1 \end{pmatrix} \;\middle|\; v_x, v_y \in \mathbb{R} \right\} \subset \mathrm{Isom}(\mathbb{R}^2)$$

is a subgroup of $\mathrm{Isom}(\mathbb{R}^2)$ consisting of all the translations and the identity.

Exercise 8. Prove that the subset

$$\left\{ \begin{pmatrix} \multicolumn{2}{c}{A} & 0 \\ & & 0 \\ 0 & 0 & 1 \end{pmatrix} \;\middle|\; A \in O(2) \right\} \subset \mathrm{Isom}(\mathbb{R}^2)$$

is a subgroup of $\mathrm{Isom}(\mathbb{R}^2)$ consisting of all the isometries that fix the origin.