COLUMBIA UNIVERSITY
IN THE CITY OF NEW YORK

Intro to Modern Algebra I
Math GU4041
New York, 2021/03/17

EXERCISE SHEET 9

Quotients, actions and semi-direct products

Exercise 1 (Conjugation of permutations)**.** Let $\sigma \in \mathfrak{S}_n$ be a k-cycle:

$$\sigma = (i_1 \;\; i_2 \;\; \cdots \;\; i_k)\,.$$

For $\tau \in \mathfrak{S}_n$, prove that

$$\tau\sigma\tau^{-1} = (\tau(i_1) \;\; \tau(i_2) \;\; \cdots \;\; \tau(i_k))\,.$$

Then, describe the conjugacy class of a general k-cycle in $\mathfrak{S}_n$. In addition, prove that the following subgroup is not normal in $\mathfrak{S}_5$:

$$\langle (1\ 2\ 3)\rangle \ntriangleleft \mathfrak{S}_5\,.$$

Exercise 2. Solve the following systems of congruences. More precisely, for every system of congruences below, find all the values of $x \in \mathbb{Z}$ that satisfy the given conditions.

(a)

$$\begin{cases} x \equiv 2 \pmod 3 \\ x \equiv 3 \pmod 5 \end{cases}$$

(b)

$$\begin{cases} x \equiv 1 \pmod 3 \\ 2x \equiv 3 \pmod 7 \end{cases}$$

(Hint: there are plenty of methods and algorithms available. But, for the very small numbers given here, you can do it easily by try and error, if you know what you are looking for. Please, consult the theorem given in class, before you start.)

Exercise 3 (Second isomorphism theorem)**.** Let G be a group, $N \triangleleft G$, $H < G$. Prove that

(a) $H \cap N \triangleleft H$.

(b) $N \triangleleft NH$.

(c) $H/(H \cap N) \simeq (NH)/N$.
(Hint: Construct a homomorphism $H \rightarrow (NH)/N$. Then apply the first isomorphism theorem.)

Exercise 4 (Third isomorphism theorem)**.** Let $\varphi : G \to G'$ be a homomorphism of G onto G'. Given $N' \lhd G'$, let $N = \varphi^{-1}(N')$. Prove that

$$G/N \simeq G'/N' .$$

Note: This is usually stated with the following suggestive formula: denote $K = \ker\varphi$, then

$$G/N \simeq (G/K)/(N/K) .$$

Exercise 5. For a group action

$$\alpha : G \times X \to X ,$$

consider the map

$$\varphi_\alpha : G \to \mathfrak{S}(X) ,$$

defined so that for every $g \in G$, $\varphi_\alpha(g)$ is the map that associates to an $x \in X$ the element $\alpha(g, x) \in X$. (This map was discussed in class). Prove that φ_α is a group homomorphism.

Exercise 6. Fix a group action $G \curvearrowright X$. Let $x, y \in X$ be points such that $O(x) = O(y)$. Prove that the stabilizer of x is conjugated to the stabilizer of y. In symbols:

$$\exists\, g \in G \ \text{ s.t. } \ \mathrm{Stab}(x) = g\ \mathrm{Stab}(y)\ g^{-1} .$$

Exercise 7. Let N, H be groups and

$$\varphi : H \to \mathrm{Aut}(N) ,$$

a homomorphism. On the set $N \times H$ consider the operation

$$(n, h)(n', h') = (n\ \varphi_h(n'), h\ h') .$$

Prove that this operation turns $N \times H$ into a group.