Intro to Modern Algebra I
Math GU4041
New York, 2021/03/24

EXERCISE SHEET 10

Finite groups

Exercise 1 (Conjugation of permutations)**.** Let $k, n \in \mathbb{N}$, $k \geq 3$ and odd, and $n \geq k$. Prove that, if $n \geq k+2$, then every two k-cycles in the alternating group A_n are conjugate. Find an example of two 3-cycles in A_4 that are not conjugate.
(Hint: Before starting, review Homework 09, Exercise 1.)

Exercise 2. For $n \in \mathbb{N}, n \geq 1$, consider the set

$$U_n = \{ \ z \in \mathbb{C} \ \mid \ z^n = 1 \ \} \subset \mathbb{C} \setminus \{0\} \, .$$

Prove that U_n is a subgroup of $\mathbb{C} \setminus \{0\}$.

Consider the group homomorphism

$$\varphi : \mathbb{R} \ni \theta \longrightarrow e^{i2\pi\theta/n} \in \mathbb{C} \setminus \{0\} \, .$$

Use the restriction of φ to $\mathbb{Z}$ to describe the structure of U_n.

The group U_n is called the group of roots of unity.

Exercise 3. Consider D_{2n}, the dihedral group (defined in class). Compute $Z(D_{2n})$.

Exercise 4. Let G be a group, and $H < G$ a subgroup such that $[G : H] = 2$. Prove that $H \lhd G$.

Exercise 5. Let $G = \mathbb{Z}_n$. Describe the set

$$\{ \ [k] \in \mathbb{Z}_n \ \mid \ \langle [k] \rangle = \mathbb{Z}_n \ \} \, .$$

of all elements that generate $\mathbb{Z}_n$.

Exercise 6. Solve the following equations

(a) $x^{13} \equiv 2 \pmod{17}$.

(b) $x^{99} \equiv 2 \pmod{20}$.

Exercise 7. Let G be a non-Abelian group. Prove that $G/Z(G)$ is not cyclic.