COLUMBIA UNIVERSITY
IN THE CITY OF NEW YORK

Intro to Modern Algebra I
Math GU4041
New York, 2021/03/31

EXERCISE SHEET 11

Cauchy's theorem

Exercise 1. Find a group G and a number d dividing $\#G$ such that G has no element of order d.

Exercise 2. Let G be a group and $a \in G$. Recall that $C(a)$ is the centralizer of A (see HW 6, Exercise 4).

(a) Prove that for all $x \in G$, $C(xax^{-1}) = xC(a)x^{-1}$.

(b) More generally, prove that if φ is an automorphism of G, then $C(\varphi(a)) = \varphi(C(a))$.

Exercise 3. Let G be a group, and $H < G$. We define the **normalizer** of H in G as

$$N(H) = \{\ x \in G \ \mid\ xHx^{-1} = H\ \}.$$

Prove the following statements:

(a) $N(H) < G$.

(b) $H \subset N(H)$.

(c) $H \lhd N(H)$.

(d) $H \lhd G \Leftrightarrow N(H) = G$.

(e) $\forall x \in G,\ N(xHx^{-1}) = xN(H)x^{-1}$.

Exercise 4. Let G be a group with subgroups A, B such that $(\#A, \#B) = 1$. Prove that

$$\#AB = \#A \cdot \#B\,.$$

Exercise 5.

(a) Prove that a group of order 99 has a nontrivial normal subgroup.

(b) Prove that a group of order 42 has a nontrivial normal subgroup.

(c) Prove that a group of order 42 has a nontrivial normal subgroup of order 21.

Exercise 6. Classify all the groups of order 55.