COLUMBIA UNIVERSITY
IN THE CITY OF NEW YORK

Intro to Modern Algebra I
Math GU4041
New York, 2021/04/07

Exercise Sheet 12

Sylow's theorem

Exercise 1. Let G be a group and $H < G$. If $a \in G$, then $aHa^{-1} < G$ is a conjugate subgroup of H in G. The set of conjugate subgroups is

$$\mathrm{Conj}(H) = \{ \ aHa^{-1} \ \mid \ a \in G \ \}.$$

Recall that $N(H)$ is the normalizer of H, from HW11, Exercise 3.

(a) Prove that $aHa^{-1} = bHb^{-1}$ if and only if $b^{-1}a \in N(H)$.

(b) Prove that, if $[G : N(H)] < \infty$, then

$$\#\mathrm{Conj}(H) = [G : N(H)].$$

(c) Prove that, if $[G : H] < \infty$, then

$$\#\mathrm{Conj}(H) \mid [G : H].$$

Exercise 2. Find the Sylow subgroups of $\mathfrak{S}_3 \times \mathfrak{S}_3$.

Exercise 3. Find a 2-Sylow subgroup of $\mathfrak{S}_5$. Then show that $n_2 = 15$.

Exercise 4. Find the Sylow subgroups of D_{12}.

Exercise 5. Prove that for every odd prime p that divides n, the Dyhedral group D_{2n} has a normal cyclic p-Sylow subgroup.

Exercise 6.

(a) Prove that a group of order 40 has a nontrivial normal subgroup.

(b) Prove that a group of order 56 has a nontrivial normal subgroup.

(c) Prove that a group of order $8p$, where p is a prime, $p > 7$ has a nontrivial normal subgroup.

Exercise 7. Let G be a group of order pqr, where $p < q < r$ are primes. Prove that G has a normal Sylow subgroup.