COLUMBIA UNIVERSITY
IN THE CITY OF NEW YORK

Number Theory and Cryptography
Math UN3020
New York, 2023/02/01

EXERCISE SHEET 3

Primes

Exercise 1 (6 points)**.** Prove that, for all $n \in \mathbb{N}$,

$$\sum_{i=0}^{2n} i^2 = \frac{n(2n+1)(4n+1)}{3}\,.$$

Exercise 2 (6 points)**.** Prove that, for all $n \in \mathbb{N}$ with $n \geq 7$,

$$n! > 3^n\,.$$

Exercise 3 (6 points)**.** Consider the sequence of real numbers defined by induction by the relations

$$\begin{cases} x_1 = 1; \\ x_{n+1} = \sqrt{1+2x_n}. \end{cases}$$

Prove that for all $n \in \mathbb{N}$ with $n \geq 1$, we have

$$x_n < 4\,.$$

Exercise 4 (9 points.)**.** Find all the solutions to each of the following Diophantine equations.

(a) $305x + 145y = 5$.

(b) $427x + 259y = 13$.

(c) $1084x + 412y = 12$.

Exercise 5 (6 points)**.** Use the Sieve of Eratosthenes to determine all primes less than 100.

Exercise 6 (6 points.)**.** Given $a, b \in \mathbb{Z}$, define

$$a' := \frac{a}{\gcd(a,b)}, \qquad b' := \frac{b}{\gcd(a,b)}\,.$$

Show that $\gcd(a', b') = 1$.

Exercise 7 (6 points)**.** To check that a given integer $n > 1$ is a prime, prove that it is enough to show that n is not divisible by any prime p with $p \leq \sqrt{n}$.

Exercise 8 (9 points)**.** Check whether the following numbers are prime.

(a) 301.

(b) 473.

(c) 1001.

Exercise 9 (6 points)**.** Let a and b two positive integers such that $a + b$ is a prime number. Prove that

$$\gcd(a, b) = 1\,.$$