Generalized stretch lines for surfaces with boundary

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Introduction. In the paper [1] we study various asymmetric metrics on the Teichmüller space of a surface S with boundary. Our goal is to extend the results obtained by Thurston for closed surfaces in his foundational paper [14]. Indeed, Thurston defines two natural distances between hyperbolic structures defined on the same topological surface, which somehow mimics the Teichmuller distance in the hyperbolic setting. The first one is the *length spectrum distance*, defined via the lengths of the simple closed curves on the surface, and the other one is the *Lipschitz distance*, defined via the Lipschitz constants of the homeomorphisms isotopic to the identity. Thurston proves that the two distances coincide by defining a preferred family of paths in the Teichmüller space, called *stretch lines*, which are geodesics for both distances. A survey with a complete exposition of Thurston's metric for closed surfaces can be found in [10]. The detailed case of the once-punctured torus was studied very recently by Dumas-Lenzhen-Rafi-Tao [4]. The convergence of the stretch lines on the boundary of the Teichmüller space was studied in various papers by Papadopoulos [9], Théret [13], Walsh [15]. Other results about the coarse geometry of the stretch lines can be found in works by Lenzhen-Rafi-Tao [6] and [7]. The analogies between Thurston's distance and Teichmüller distance were studied by Choi-Rafi [3].

While it is well-known that Thurston's stretch lines are not the only geodesics, the properties of the geodesic flow of the Thurston metric are still rather mysterious. In the case of closed surfaces the length spectrum distance is the key ingredient in the construction of the stretch lines. In the case of surfaces with boundary, Parlier [12] proved that Thurston's length spectrum distance is no longer a distance as it can be negative. Guéritaud-Kassel [5] studied this functional in detail in the case when the surface has funnels. They also give beautiful applications of their work to the affine actions on $\mathbb{R}^3$ and Margulis space-times. When the surface has compact boundary, there are many asymmetric distances that mimic Thurston's distance for closed surfaces: the *arc distance* d_A, defined via the lengths of simple arcs and curves on S (first defined by Liu-Papadopoulos-Su-Théret [8]); the *Lipschitz distance* d_{Lh}, defined via the Lipschitz constants of the homeomorphisms isotopic to the identity (as defined by Thurston [14] for closed surfaces); the *Lipschitz map distance* $d_{L\partial}$, defined via the Lipschitz constants of the maps isotopic to the identity (we will define it here). It is immediate to see that $d_A \leq d_{L\partial} \leq d_{Lh}$.

Question 1. *Do Thurston's results still extend to the case of surfaces with boundary for some (or all) the distances above? Do the distances* $d_A, d_{L\partial}$ *and* d_{Lh} *coincide?*

Denote by S^d the *double* of S, that is, the surface obtained glueing two copies of S the boundary. Furthermore, denote by X^d the hyperbolic structure on S^d

obtained doubling the hyperbolic structure X on S. There is a natural embedding:

$$\mathrm{Teich}(S) \ni X \hookrightarrow X^d \in \mathrm{Teich}(S^d).$$

Question 2. *Is the natural embedding* $(\mathrm{Teich}(S), d_?) \hookrightarrow (\mathrm{Teich}(S^d), d_{Th})$ *a geodesic embedding for* $d_? = d_A, d_{L\partial}$ *or* d_{Lh}*?*

In our paper [1] we investigate these questions for d_A and d_{Lh}.

Our results. We will first construct a large family of paths in Teich(S) that we will call *generalized stretch lines* following Thurston [14]. We will prove that our generalized stretch lines are also geodesics for the two distances d_A and $d_{L\partial}$. As in Thurston [14], for any two points on the same generalized stretch line, there is a special Lipschitz map between them having optimal Lipschitz constant. We will call this map a *generalized stretch map.*

Theorem 3 (Existence of generalized stretch maps)**.** *Let S be a surface with non-empty boundary and let X be a hyperbolic structure on S. For every maximal lamination λ on S and for every $t \geq 0$ there exists a hyperbolic structure X_λ^t and a Lipschitz map $\Phi^t : (S, X) \longrightarrow (S, X_\lambda^t)$ with the following properties:*

1. $\Phi^t(\partial S) = \partial S$;
2. Φ^t *stretches the arc length of the leaves of λ by the factor e^t;*
3. *for every geometric piece $\mathcal{G}$ in $S \setminus \lambda$ the map Φ^t restricts to a generalized stretch map $\phi^t : \mathcal{G} \longrightarrow \mathcal{G}_t$;*
4. $\mathrm{Lip}(\Phi^t) = e^t$

Furthermore, if λ contains a non-empty measurable sublamination we have:

$$\mathrm{Lip}(\Phi^t) = \min\{\mathrm{Lip}(\psi) \mid \psi \in \mathrm{Lip}_0(X, X_\lambda^t), \psi(\partial S) \subset \partial S\}.$$

In the case of [14], any maximal lamination decomposes a closed surface into ideal triangles and Thurston uses the horocycle foliation to construct explicitely the stretch map between two ideal triangles. In the case of surfaces with boundary, new problems arise. Indeed, a maximal lamination decomposes a surface with boundary in geometric pieces of four different types: ideal triangles; right-angled quadrilaterals with two consecutive ideal vertices; right-angled pentagons with one ideal vertex, and right-angled hexagons. Unlike Thurston [14], we will not construct the maps explicitely.

Corollary 4 (Existence of generalized stretch lines)**.** *The path*

$$s_\lambda : \mathbb{R}_{\geq 0} \longrightarrow \mathrm{Teich}(S)$$
$$t \mapsto X_\lambda^t$$

is a geodesic path parametrized by arc-length for both d_A and $d_{L\partial}$.

We will call the path s_λ a *generalized stretch line.*

Corollary 5. *For any two hyperbolic structures X, Y on S, there exists a continuous map $\phi \in \mathrm{Lip}_0(X, Y)$, with $\phi(\partial S) \subset \partial S$ such that*

$$\log(\mathrm{Lip}(\phi)) = d_A(X, Y).$$

This map has an optimal Lipschitz constant. Therefore, $d_A = d_{L\partial}$.

We use Theorem 3 to prove an analogue of Thurston's theorem.

Theorem 6. *The space* $(\mathrm{Teich}(S), d_A)$ *is a geodesic metric space. Furthermore, any two points* $X, Y \in \mathrm{Teich}(S)$ *can be joined by a geodesic for* d_A *and* $d_{L\partial}$*, which is a finite concatenation of generalized stretch lines.*

Corollary 7. *There exists a Finsler metric on* $\mathrm{Teich}(S)$ *whose induced distance is* d_A*.*

Our results also have applications in the case of closed surfaces. Indeed, we can answer positively to Question 2. Our proof relies on a result by Liu-Papadopoulos-Théret-Su [8].

Corollary 8. *The embedding* $\jmath : (\mathrm{Teich}(S), d_A) \hookrightarrow (\mathrm{Teich}(S^d), d_{Th})$ *is a geodesic embedding, i.e. every two points in the image are joined by a geodesic in the image.*

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Computations on Johnson homomorphisms

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Let $\mathcal{M}_g$ be the mapping class group of a closed oriented surface Σ_g of genus g and let $\mathcal{I}_g \subset \mathcal{M}_g$ be the Torelli subgroup. Namely, it is the subgroup of $\mathcal{M}_g$ consisting of all the elements which act on the homology $H_1(\Sigma_g;\mathbb{Z})$ trivially.

There exist two filtrations of the Torelli group. One is the lower central series which we denote by $\mathcal{I}_g(k)$ $(k = 1, 2, \ldots)$ where $\mathcal{I}_g(1) = \mathcal{I}_g$ and $\mathcal{I}_g(k+1) = [\mathcal{I}_g(k), \mathcal{I}_g]$ for $k \geq 1$. The other is called the Johnson filtration $\mathcal{M}_g(k)$ $(k = 1, 2, \ldots)$ of the mapping class group where $\mathcal{M}_g(k)$ is defined to be the kernel of the natural homomorphism

$$\rho_k : \mathcal{M}_g \to \mathrm{Out}(N_k(\pi_1\Sigma_g)).$$

Here $N_k(\pi_1\Sigma_g)$ denotes the k-th nilpotent quotient of the fundamental group of Σ_g and $\mathrm{Out}(N_k(\pi_1\Sigma_g))$ denotes its outer automorphism group. $\mathcal{M}_g(1)$ is nothing other than the Torelli group $\mathcal{I}_g$ so that $\mathcal{M}_g(k)$ $(k = 1, 2, \ldots)$ is a filtration of $\mathcal{I}_g$. This filtration was originally introduced by Johnson [7] for the case of a genus g surface with one boundary component. The above is the one adapted to the case of a closed surface. It can be shown that $\mathcal{I}_g(k) \subset \mathcal{M}_g(k)$ for all $k \geq 1$. Johnson showed in [9] that $\mathcal{I}_g(2)$ is a finite index subgroup of $\mathcal{M}_g(2)$ and asked whether this will continue to hold for the pair $\mathcal{I}_g(k) \subset \mathcal{M}_g(k)$ $(k \geq 3)$. He also showed in [8] that $\mathcal{M}_g(2)$ is equal to the subgroup $\mathcal{K}_g$, which is called the Johnson subgroup or Johnson kernel, consisting of all the Dehn twists along separating simple closed curves on Σ_g.

The above question was answered negatively in [10]. More precisely, a homomorphism $d_1 : \mathcal{K}_g \to \mathbb{Z}$ was constructed which is non-trivial on $\mathcal{M}_g(3)$ while it vanishes on $\mathcal{I}_g(3)$ so that the index of the pair $\mathcal{I}_g(3) \subset \mathcal{M}_g(3)$ was proved to be infinite. Furthermore it was shown in [11] that there exists an isomorphism

$$H^1(\mathcal{K}_g;\mathbb{Z})^{\mathcal{M}_g} \cong \mathbb{Z} \quad (g \geq 2)$$

where the homomorphism d_1 serves as a rational generator. It is characterized by the fact that its value on a separating simple closed curve on Σ_g of type $(h, g-h)$ is $h(g-h)$ up to non-zero constants. This homomorphism was defined as the secondary characteristic class associated with the fact that the first MMM class, which is an element of $H^2(\mathcal{M}_g;\mathbb{Z})$, vanishes in $H^2(\mathcal{I}_g;\mathbb{Z})$.