

Logarithmic limit sets of real semi-algebraic sets

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Abstract. This paper is about the logarithmic limit sets of real semi-algebraic sets, and, more generally, about the logarithmic limit sets of sets definable in an o-minimal, polynomially bounded structure. We prove that most of the properties of the logarithmic limit sets of complex algebraic sets hold in the real case. This includes the polyhedral structure and the relation with the theory of non-Archimedean fields, tropical geometry and Maslov dequantization.

1 Introduction

Logarithmic limit sets of complex algebraic sets have been studied extensively. They first appeared in Bergman's paper [3], and then they were further studied by Bieri and Groves in [4]. Recently their relations with the theory of non-Archimedean fields and tropical geometry were discovered (see for example [17], [10] and [6]). They are now usually called tropical varieties, but they appeared also under the names of Bergman fans, Bergman sets, Bieri-Groves sets or non-Archimedean amoebas. The logarithmic limit set of a complex algebraic set is a polyhedral complex of the same dimension as the algebraic set, it is described by tropical equations and it is the image of an algebraic set over an algebraically closed non-Archimedean field under the component-wise valuation map. The tools used to prove these facts are mainly algebraic and combinatorial.

In this paper we extend these results to the logarithmic limit sets of real algebraic and semi-algebraic sets. The techniques we use to prove these results in the real case are very different from the ones used in the complex case. Our main tool is the cell decomposition theorem, as we prefer to look directly at the geometric set, instead of using its equations. In the real case, even if we restrict our attention to an algebraic set, it seems that the algebraic and combinatorial properties of the defining equations do not give enough information to study the logarithmic limit set.

In the following we often need to act on $(\mathbb{R}_{>0})^n$ with maps of the form

$$\varphi_A(x_1, \dots, x_n) = (x_1^{a_{11}} \cdots x_n^{a_{1n}}, x_1^{a_{21}} \cdots x_n^{a_{2n}}, \dots, x_1^{a_{n1}} \cdots x_n^{a_{nn}}),$$

where $A = (a_{ij})$ is an $n \times n$ matrix. When the entries of A are not rational, the image of a semi-algebraic set is, in general not semi-algebraic. Actually, the only thing we can say about images of semi-algebraic sets via these maps is that they are definable in the structure of the real field expanded with arbitrary power functions. This structure, usually denoted by $\overline{\mathbb{R}^{\mathbb{R}}}$, is o-minimal and polynomially bounded, and these are the main properties we need in the proofs. Moreover, if S is a set definable in $\overline{\mathbb{R}^{\mathbb{R}}}$, then the image $\varphi_A(S)$ is again definable, as the functions $x \rightarrow x^\alpha$ are definable for every α . If a structure has the property that all power functions with every real exponent are definable, that structure is said to have field of exponents \mathbb{R} .

In this sense the category of semi-algebraic sets is too small for our methods. It seems that the natural context for the study of logarithmic limit sets is to fix a general expansion of the structure of the real field that is o-minimal and polynomially bounded, with field of exponents \mathbb{R} . For sets definable in such a structure, the properties that were known for the complex algebraic sets also hold. We can prove that these logarithmic limit sets are polyhedral complexes with dimension less than or equal to the dimension of the definable set.

Then we show how the relation between tropical varieties and images of varieties defined over non-Archimedean fields, well known for algebraically closed fields, can be extended to the case of real closed fields. We give the notion of non-Archimedean amoebas of semi-algebraic sets over non-Archimedean fields, and also for sets definable in other o-minimal structures on non-Archimedean fields. We show that the logarithmic limit set of a definable subset of \mathbb{R}^n is the non-Archimedean amoeba of an extension of the definable set to a real closed non-Archimedean field. We also study the relations between non-Archimedean amoebas and patchworking families of definable sets. Note that this notion of non-Archimedean amoebas of definable sets generalizes the notion of non-Archimedean amoebas of semi-linear sets that has been used in [8] to study tropical polytopes.

An analysis of the defining equations and inequalities is carried out, showing that the logarithmic limit set of a closed semi-algebraic set can be described applying the Maslov dequantization to a suitable formula defining the semi-algebraic set.

Our motivation for this work comes from the study of Teichmüller spaces and, more generally, of spaces of geometric structures on manifolds. In the papers [1] and [2] we present a construction of compactifications using the logarithmic limit sets. The properties of logarithmic limit sets we prove here will be used in [1] to describe the compactification. For example, the fact that logarithmic limit sets of real semi-algebraic sets are polyhedral complexes will provide an independent construction of the piecewise linear structure on the Thurston boundary of Teichmüller spaces. Moreover the relations with tropical geometry and the theory of non-Archimedean fields will be used in [2] for constructing a geometric interpretation of the boundary points.

In Section 2 we define logarithmic limit sets for general subsets of $(\mathbb{R}_{>0})^n$, and we recall some preliminary notions of model theory and o-minimal geometry that we will use in the following, most notably the notion of regular polynomially bounded structures. In Section 3 we prove that logarithmic limit sets of definable sets in a regular polynomially bounded structure are polyhedral complexes with dimension less than or equal to the

dimension of the definable set, and we provide a local description of these sets. The main tool we use in this section is the cell decomposition theorem.

In Section 4 we consider a special class of non-Archimedean fields: the Hardy fields of regular polynomially bounded structures. These are non-Archimedean real closed fields of rank one extending \mathbb{R} , with a canonical real valued valuation and residue field \mathbb{R} . The elements of these fields are germs of definable functions, hence they have better geometric properties than the fields of formal series usually employed in tropical geometry. The images, under the component-wise valuation map, of definable sets in the Hardy fields are related with the logarithmic limit sets of real definable sets, and with the limit of real patchworking families.

In Section 5 we compare the construction of this paper with other known constructions. We show that the logarithmic limit sets of complex algebraic sets are only a particular case of the logarithmic limit sets of real semi-algebraic sets, and the same happens for the limit of complex patchworking families. Hence our methods provide an alternative proof (with a topological flavor) for some known results about complex sets. We also compare the logarithmic limit sets of real algebraic sets with the construction of Positive Tropical Varieties (see [18]). Even if in many examples these two notions coincide, we show some examples where they differ.

In Section 6 we show how the construction of Maslov dequantization provides a relation between logarithmic limit sets of semi-algebraic sets and the tropical semifield. We show that for every closed semi-algebraic set there exists a defining formula such that the tropicalization of that formula defines the logarithmic limit set. This result is the analog, for real semi-algebraic sets, of the existence of a tropical basis for complex algebraic sets.

2 Preliminaries

2.1 Notation. If $x \in \mathbb{R}^n$ we will denote its coordinates by x_1, \dots, x_n . If $\omega \in \mathbb{N}^n$ we will use the multi-index notation for powers: $x^\omega = x_1^{\omega_1} \dots x_n^{\omega_n}$. We will consider also powers with real exponents, if the base is positive, hence if $x \in (\mathbb{R}_{>0})^n$ and $\omega \in \mathbb{R}^n$ we will write $x^\omega = x_1^{\omega_1} \dots x_n^{\omega_n}$.

If $(s(k))$ is a sequence in \mathbb{R}^n , we will denote its k -th element by $s(k) \in \mathbb{R}^n$. The subscript notation will be reserved for the coordinates $s_i(k) \in \mathbb{R}$.

Given a real number $\alpha > 1$, we will denote by Log_α the *component-wise logarithm map*, and by Exp_α its inverse:

$$\begin{aligned} \text{Log}_\alpha : (\mathbb{R}_{>0})^n \ni (x_1, \dots, x_n) &\longrightarrow (\log_\alpha(x_1), \dots, \log_\alpha(x_n)) \in \mathbb{R}^n \\ \text{Exp}_\alpha : \mathbb{R}^n \ni (x_1, \dots, x_n) &\longrightarrow (\alpha^{x_1}, \dots, \alpha^{x_n}) \in (\mathbb{R}_{>0})^n \end{aligned}$$

We define a notion of *limit* for every one-parameter family of subsets of \mathbb{R}^n . Suppose that for all $t \in (0, \varepsilon)$ we have a set $S_t \subset \mathbb{R}^n$. We can construct the deformation

$$\mathcal{D}(S \cdot) = \{(x, t) \in \mathbb{R}^n \times (0, \varepsilon) \mid x \in S_t\}.$$

We denote by $\overline{\mathcal{D}(S \cdot)}$ the closure of $\mathcal{D}(S \cdot)$ in $\mathbb{R}^n \times [0, \varepsilon)$, then we define

$$\lim_{t \rightarrow 0} S_t = \pi(\overline{\mathcal{D}(S \cdot)}) \cap \mathbb{R}^n \times \{0\} \subset \mathbb{R}^n,$$

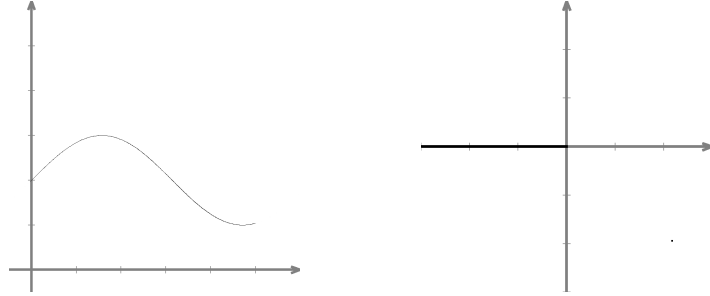


Figure 1. If $V = \{(x, y) \in (\mathbb{R}_{>0})^2 \mid y = \sin x + 2, x \leq 5\}$ (left picture), then $\mathcal{A}_0(V) = \{(x, y) \in \mathbb{R}^2 \mid y = 0, x \leq 0\}$ (right picture).

where $\pi : \mathbb{R}^n \times [0, \varepsilon) \rightarrow \mathbb{R}^n$ is the projection on the first factor. This limit is well defined for every family of subsets of \mathbb{R}^n .

Proposition 2.1. *The set $S = \lim_{t \rightarrow 0} S_t$ is a closed subset of \mathbb{R}^n . A point y is in S if and only if there exist a sequence $(y(k))$ in \mathbb{R}^n and a sequence $(t(k))$ in $(0, \varepsilon)$ such that $(t(k)) \rightarrow 0$, $(y(k)) \rightarrow y$ and $\forall k \in \mathbb{N} : y(k) \in S_{t(k)}$.*

2.2 Logarithmic limit sets of general sets. Given a set $V \subset (\mathbb{R}_{>0})^n$ and a number $t \in (0, 1)$, the amoeba of V is

$$\mathcal{A}_t(V) = \text{Log}_{\frac{1}{t}}(V) = \frac{-1}{\log_e(t)} \text{Log}_e(V) \subset \mathbb{R}^n.$$

The limit of the amoebas (in the sense of subsection 2.1) is the *logarithmic limit set* of V :

$$\mathcal{A}_0(V) = \lim_{t \rightarrow 0} \mathcal{A}_t(V).$$

Some examples of logarithmic limit sets are in Figures 1 and 2.

Proposition 2.2. *Given a set $V \subset (\mathbb{R}_{>0})^n$ the following properties hold:*

- (1) *The logarithmic limit set $\mathcal{A}_0(V)$ is closed and $y \in \mathcal{A}_0(V)$ if and only if there exist a sequence $(x(k))$ in V and a sequence $(t(k))$ in $(0, 1)$ such that $(t(k)) \rightarrow 0$ and*

$$\left(\text{Log}_{\frac{1}{t(k)}}(x(k)) \right) \rightarrow y.$$

- (2) *The logarithmic limit set $\mathcal{A}_0(V)$ is a cone in \mathbb{R}^n .*
 (3) *We have $0 \in \mathcal{A}_0(V)$ if and only if $V \neq \emptyset$. Moreover, $\mathcal{A}_0(V) = \{0\}$ if and only if V is relatively compact in $(\mathbb{R}_{>0})^n$ and non-empty.*
 (4) *If $W \subset \mathbb{R}^n$ we have $\mathcal{A}_0(V \cup W) = \mathcal{A}_0(V) \cup \mathcal{A}_0(W)$ and $\mathcal{A}_0(V \cap W) \subset \mathcal{A}_0(V) \cap \mathcal{A}_0(W)$.*

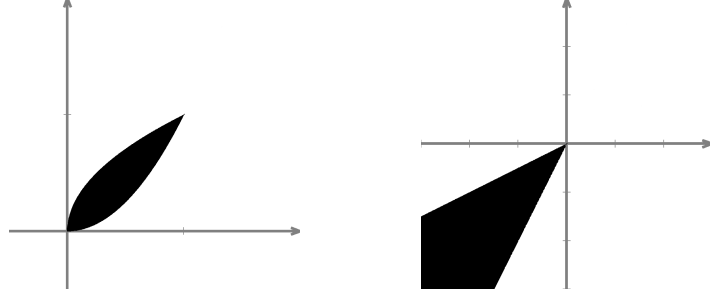


Figure 2. If $V = \{(x, y) \in (\mathbb{R}_{>0})^2 \mid x^2 \leq y \leq \sqrt{x}\}$ (left picture), then $\mathcal{A}_0(V) = \{(x, y) \in \mathbb{R}^2 \mid 2x \leq y \leq \frac{1}{2}x\}$ (right picture).

Proof. The first assertion is simply a restatement of Proposition 2.1. For the second one, we want to prove that if $\lambda > 0$ and $y \in \mathcal{A}_0(V)$, then $\lambda^{-1}y \in \mathcal{A}_0(V)$. There exist a sequence $(x(k))$ in V and a sequence $(t(k))$ in $(0, 1)$ such that $(t(k)) \rightarrow 0$ and $(\text{Log}_{\frac{1}{t(k)}}(x(k))) \rightarrow y$. Consider the sequences $(x(k))$ and $(t(k)^\lambda)$. Now

$$\text{Log}_{\frac{1}{t(k)^\lambda}}(x(k)) = \frac{-1}{\log_e(t(k)^\lambda)} \text{Log}_e(x(k)) = \lambda^{-1} \frac{-1}{\log_e(t(k))} \text{Log}_e(x(k))$$

and this sequence converges to $\lambda^{-1}y$. The third and fourth assertions are trivial. \square

Given a closed cone $C \subset \mathbb{R}^n$, there always exists a set $V \subset (\mathbb{R}_{>0})^n$ such that $C = \mathcal{A}_0(V)$, simply take $V = \text{Log}_e^{-1}(C)$. Then $\mathcal{A}_t(V) = C$ for all t . If we want to find more properties of logarithmic limit sets, we need some assumptions on the set V , for example that V is semi-algebraic or, more generally, definable in an o-minimal polynomially bounded structure, as in Section 3.

Let $A = (a_{ij}) \in \text{GL}_n(\mathbb{R})$. The matrix A acts on \mathbb{R}^n in the natural way and, via conjugation with the map Log_e , it acts on $(\mathbb{R}_{>0})^n$. Explicitly, it induces the maps $A : \mathbb{R}^n \rightarrow \mathbb{R}^n$ and $\bar{A} : (\mathbb{R}_{>0})^n \rightarrow (\mathbb{R}_{>0})^n$:

$$\begin{aligned} A(x) &= A(x_1, \dots, x_n) = (a_{11}x_1 + \dots + a_{1n}x_n, \dots, a_{n1}x_1 + \dots + a_{nn}x_n) \\ \bar{A}(x) &= \text{Exp}_e \circ A \circ \text{Log}_e(x) = (x_1^{a_{11}} x_2^{a_{12}} \dots x_n^{a_{1n}}, \dots, x_1^{a_{n1}} x_2^{a_{n2}} \dots x_n^{a_{nn}}) \end{aligned}$$

If $V \subset (\mathbb{R}_{>0})^n$ and $B \in \text{GL}_n(\mathbb{R})$, then $B(\mathcal{A}_0(V)) = \mathcal{A}_0(\bar{B}(V))$.

Lemma 2.3. $(0, \dots, 0, -1) \in \mathcal{A}_0(V)$ if and only if there exists a sequence $(y(k))$ in V such that $(y_n(k)) \rightarrow 0$ and

$$\begin{aligned} \forall N \in \mathbb{N} : \exists k_0 \in \mathbb{N} : \forall k > k_0 : \forall i \in \{1, \dots, n-1\} : \\ y_n(k) < (y_i(k))^N \text{ and } y_n(k) < (y_i(k))^{-N}. \end{aligned}$$

Proof. Suppose $(0, \dots, 0, -1) \in \mathcal{A}_0(V)$, then by Proposition 2.2 there exists a sequence $(y(k))$ in V and a sequence $(t(k))$ in $(0, 1)$ such that $(t(k)) \rightarrow 0$ and $(\text{Log}_{\frac{1}{t(k)}}(y(k))) \rightarrow (0, \dots, 0, -1)$. This means that

$$\left(\frac{-1}{\log_e(t(k))} \log_e(y_i(k)) \right) \rightarrow \begin{cases} -1 & \text{if } i = n \\ 0 & \text{if } i \in \{1, \dots, n-1\} \end{cases}$$

Now $(t(k)) \rightarrow 0$ hence $(\frac{-1}{\log_e(t(k))}) \rightarrow 0$, $(\log_e(y_n(k))) \rightarrow -\infty$, $(y_n(k)) \rightarrow 0$ and

$$\forall i \in \{1, \dots, n-1\} : \left(\frac{\log_e(y_i(k))}{\log_e(y_n(k))} \right) \rightarrow 0.$$

Hence $\forall \varepsilon > 0 : \exists k_0 : \forall k > k_0 : \forall i < n : (y_n(k))^\varepsilon < y_i(k) < (y_n(k))^{-\varepsilon}$. We conclude by reversing the inequalities and choosing $\varepsilon = \frac{1}{N}$.

Conversely, if $(y(k))$ has the stated property, then $(|\text{Log}_e(y(k))|) \rightarrow \infty$ (where $|\cdot|$ is the standard Euclidean norm in \mathbb{R}^n). It is possible to choose $(t(k))$ such that $(t(k)) \rightarrow 0$ and $|\text{Log}_{\frac{1}{t(k)}}(y(k))| = 1$. Up to subsequences, the sequence $(\text{Log}_{\frac{1}{t(k)}}(y(k)))$ converges to a point that, by reversing the calculations in first part of the proof, is $(0, \dots, 0, -1)$. Hence $(0, \dots, 0, -1) \in \mathcal{A}_0(V)$. \square

Corollary 2.4. *If there exists a sequence $(x(k))$ in V such that $(x(k)) \rightarrow (a_1, \dots, a_{n-1}, 0)$, where $a_1, \dots, a_{n-1} > 0$, then $(0, \dots, 0, -1) \in \mathcal{A}_0(V)$. The converse is not true in general (see Figure 3).*

We will see in Theorem 3.6 that if V is definable in an o-minimal and polynomially bounded structure, the converse of the corollary becomes true.

A sequence $(b(k))$ in $(\mathbb{R}_{>0})^n$ is in *standard position* in dimension m if, for $g = n-m$, we have $(b(k)) \rightarrow b = (b_1, \dots, b_g, 0, \dots, 0)$, with $b_1, \dots, b_g > 0$ and

$$\forall N \in \mathbb{N} : \exists k_0 : \forall k > k_0 : \forall i \in \{g+1, \dots, n-1\} : b_{i+1}(k) < (b_i(k))^N.$$

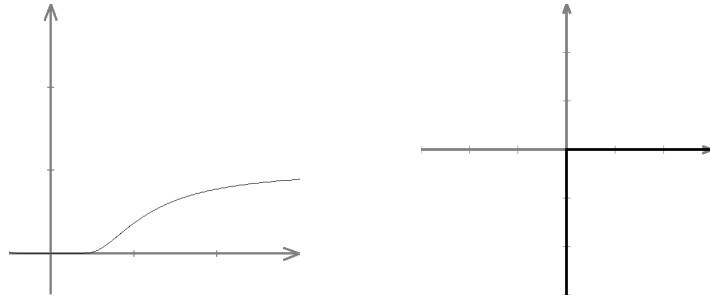


Figure 3. If $V = \{(x, y) \in (\mathbb{R}_{>0})^2 \mid y = e^{-\frac{1}{x^2}}\}$ (left picture), then $\mathcal{A}_0(V) = \{(x, y) \in \mathbb{R}^2 \mid y = 0, x \geq 0 \text{ or } x = 0, y \leq 0\}$ (right picture).

Lemma 2.5. *Let $(a(k))$ be a sequence in $(\mathbb{R}_{>0})^n$ such that $(a(k)) \rightarrow a = (a_1, \dots, a_h, 0, \dots, 0)$, with $h < n$ and $a_1, \dots, a_h > 0$. There exist a number $m \leq n - h$, a subsequence (again denoted by $(a(k))$) and a linear map $A : \mathbb{R}^n \rightarrow \mathbb{R}^n$ such that the sequence $(b(k)) = (\overline{A}(a(k))) \subset (\mathbb{R}_{>0})^n$ is in standard position in dimension m .*

Proof. By induction on n . For $n = 1$ the statement is trivial. Suppose that the statement holds for $n - 1$. Consider the logarithmic image of the sequence: $(\text{Log}_e(a(k)))$. Up to extracting a subsequence, the sequence $(\frac{\text{Log}_e(a(k))}{|\text{Log}_e(a(k))|})$ converges to a unit vector $v = (0, \dots, 0, v_{h+1}, \dots, v_n)$. There exists a linear map B , acting only on the last $n - h$ coordinates, sending v to $(0, \dots, 0, -1)$. By Lemma 2.3, the map \overline{B} sends $(a(k))$ to a sequence $(b(k))$ such that $(b_n(k)) \rightarrow 0$ and $\forall N \in \mathbb{N} : \exists k_0 : \forall k > k_0 : \forall i \in \{1, \dots, n - 1\}$:

$$b_n(k) < (b_i(k))^N \quad \text{and} \quad b_n(k) < (b_i(k))^{-N} \quad (1)$$

As B acts only on the last $n - h$ coordinates, $(b_i(k)) \rightarrow a_i \neq 0$ for $i \in \{1, \dots, h\}$. Up to subsequences we can suppose that for every $i \in \{h + 1, \dots, n - 1\}$ one of the following three possibilities occurs: $(b_i(k)) \rightarrow 0$, $(b_i(k)) \rightarrow b_i \neq 0$, $(b_i(k)) \rightarrow +\infty$. Up to a change of coordinates with maps of the form

$$\begin{aligned} B_i(x_1, \dots, x_i, \dots, x_n) &= (x_1, \dots, -x_i, \dots, x_n) \\ \overline{B}_i(x_1, \dots, x_i, \dots, x_n) &= (x_1, \dots, x_i^{-1}, \dots, x_n), \end{aligned}$$

we can suppose that for every $i \in \{h + 1, \dots, n - 1\}$ either $(b_i(k)) \rightarrow 0$ or $(b_i(k)) \rightarrow b_i \neq 0$. Up to reordering the coordinates, we can suppose that there exists $g \geq h$ such that for $i \in \{1, \dots, g\}$: $(b_i(k)) \rightarrow b_i \neq 0$ and for $i > g$, $(b_i(k)) \rightarrow 0$. Now consider the projection on the first $n - 1$ coordinates: $\pi : \mathbb{R}^n \rightarrow \mathbb{R}^{n-1}$. By the inductive hypothesis there exists a linear map $C : \mathbb{R}^{n-1} \rightarrow \mathbb{R}^{n-1}$ sending the sequence $(\pi(b(k)))$ to a sequence $(c(k))$ satisfying

$$\forall N \in \mathbb{N} : \exists k_0 : \forall k > k_0 : \forall i \in \{g + 1, \dots, n - 2\} : c_{i+1}(k) < (c_i(k))^N \quad (2)$$

The composition of B and a map that preserves the last coordinate and acts as C on the first ones is the linear map we are searching for. For $i \in \{g + 1, \dots, n - 2\}$, the inequality follows from (2), for $i = n - 1$, it follows from (1). \square

The *basic cone* defined by the vector $N = (N_1, \dots, N_{n-1}) \in \mathbb{N}^{n-1}$ is

$$B_N = \{x \in \mathbb{R}^n \mid \forall i : x_i \leq 0 \text{ and } \forall i < n : x_{i+1} \leq N_i x_i\}.$$

Note that if $N' = (N'_1, \dots, N'_{n-1})$, with $\forall i : N'_i \geq N_i$, then $B'_{N'} \subset B_N$. The *exponential basic cone* in $(\mathbb{R}_{>0})^n$ defined by the vector $N = (N_1, \dots, N_{n-1}) \in \mathbb{N}^{n-1}$ and the scalar $h > 0$ is the set

$$E_{N,h} = \{x \in \mathbb{R}^n \mid \forall i : 0 < x_i \leq h \text{ and } \forall i < n : x_{i+1} \leq x_i^{N_i}\}.$$

Lemma 2.6. *The following easy facts about basic cones hold:*

(1) *The logarithmic limit set of an exponential basic cone is a basic cone:*

$$\mathcal{A}_0(E_{N,h}) = B_N.$$

(2) *If $(b(k))$ is a sequence in $(\mathbb{R}_{>0})^n$ in standard position in dimension n , and $E_{N,h}$ is an exponential basic cone, then for large enough k , $b(k) \in E_{N,h}$.*

2.3 Definable sets in o-minimal structures. In this subsection we recall notation and some definitions of model theory and o-minimal geometry we will use later, see [9] and [19] for details. Given a *set of symbols* S (see [9, Chapter II, Definition 2.1]), we denote by L_S the corresponding *first order language* (see [9, Chapter II, Definition 3.2]). If S' is an *expansion* of a set of symbols S we will write $S \subset S'$. For example, for the theory of real closed fields one can use the set of symbols of ordered semirings: $\mathcal{OS} = (\{\leq\}, \{+, \cdot, \emptyset\})$ or the set of symbols of ordered rings $\mathcal{OR} = (\{\leq\}, \{+, -, \cdot\}, \{0, 1\})$, which is an expansion of \mathcal{OS} . In the following we will use these sets of symbols and some of their expansions.

We usually will denote an S -structure by $\overline{M} = (M, a)$, where M is a set, and a is the *interpretation* (see [9, Chapter III, Definition 1.1]). Given an S -structure $\overline{M} = (M, a)$, and an L_S -formula φ without free variables, we will write $\overline{M} \models \varphi$ if \overline{M} satisfies φ (see [9, Chapter III, Definition 3.1]).

A real closed field can be defined as an \mathcal{OS} - or an \mathcal{OR} -structure satisfying a suitable infinite set of first order axioms. The natural \mathcal{OS} -structure on \mathbb{R} will be denoted by $\overline{\mathbb{R}}$.

If $\overline{M} = (M, a)$ is an S -structure, and S' is an expansion of S , an S' -structure (M, a') is an *expansion* of the S -structure (M, a) if a' restricted to the symbols of S is equal to a . If $M \subset N$, an S -structure $\overline{N} = (N, b)$ is an *extension* of an S -structure $\overline{M} = (M, a)$ if for all $s \in M$, $b(s)|_M = a(s)$.

A *definable subset* of M^n is a set that is defined by an L_S -formula $\varphi(x_1, \dots, x_n, y_1, \dots, y_m)$ and by parameters $a_1, \dots, a_m \in M$, and a *definable map* is a map whose graph is definable. For example if M is an \mathcal{OS} - or an \mathcal{OR} -structure satisfying the axioms of real closed fields, the definable sets are the semi-algebraic sets, and the definable maps are the semi-algebraic maps.

Let S be an expansion of \mathcal{OS} , and let \overline{M} be an S -structure satisfying the axioms of the real closed fields. The S -structure \overline{M} is said to be *o-minimal* if the definable subsets of M are the finite unions of points and intervals (bounded and unbounded). See also [7] and [19].

Let S be an expansion of \mathcal{OS} , and let $\mathfrak{R} = (\mathbb{R}, a)$ be an o-minimal S -structure. The S -structure \mathfrak{R} is called *polynomially bounded* if for every definable function $f : \mathbb{R} \rightarrow \mathbb{R}$ there exists a natural number N such that for every sufficiently large x , $|f(x)| \leq x^N$. See also [12]. In [13] it is shown that, in this case, if $f : \mathbb{R} \rightarrow \mathbb{R}$ is definable and not ultimately 0, there exist $r, c \in \mathbb{R}$, $c \neq 0$, such that

$$\lim_{x \rightarrow +\infty} \frac{f(x)}{x^r} = c.$$

The set of all such r is a subfield of \mathbb{R} , called the *field of exponents* of \mathfrak{R} . For example the \mathcal{OR} -structure $\overline{\mathbb{R}}$ is polynomially bounded with field of exponents \mathbb{Q} .

If $\Lambda \subset \mathbb{R}$ is a subfield, we can construct an expansion of S and \mathfrak{A} by adding the power functions with exponents in Λ . We expand S to S^Λ by adding a function symbol f_λ for every $\lambda \in \Lambda$, and we expand \mathfrak{A} to an S^Λ -structure \mathfrak{A}^Λ interpreting the function symbol f_λ by the function that is $x \rightarrow x^\lambda$ for positive numbers and $x \rightarrow 0$ on negative ones. The structure \mathfrak{A}^Λ is again o-minimal, as its definable sets are definable in the structure \mathfrak{A} expanded by adding the exponential function e^x , which is o-minimal by [16].

Suppose that the expansion of \mathfrak{A} constructed by adding the family of functions $\{x^r_{|[1,2]}\}_{r \in \Lambda}$ is polynomially bounded, then \mathfrak{A}^Λ is too (see [14]). For example if the structure \mathfrak{A} expanded by adding the restricted exponential function $e^x_{|[0,1]}$ is polynomially bounded, then \mathfrak{A}^Λ is too.

In the following we will work with o-minimal, polynomially bounded structures \mathfrak{A} expanding $\overline{\mathbb{R}}$, with the property that $\mathfrak{A}^{\mathbb{R}}$ is polynomially bounded. We will call such structures *regular polynomially bounded structures*. One example of regular polynomially bounded structure is \mathbb{R}_{an} , the real numbers with restricted analytic functions, see [20] for details. This structure has field of exponents \mathbb{Q} , while $\mathbb{R}_{\text{an}}^{\mathbb{R}}$ has field of exponents \mathbb{R} . As \mathbb{R}_{an} is an expansion of $\overline{\mathbb{R}}$, also $\overline{\mathbb{R}}^{\mathbb{R}}$ is polynomially bounded, hence $\overline{\mathbb{R}}$ is a regular polynomially bounded structure.

Other examples of regular polynomially bounded structures are the structure \mathbb{R}_{an^*} of the real field with convergent generalized power series (see [22]), the field of real numbers with multisummable series (see [23]) and the structures defined by a quasianalytic Denjoy–Carleman class (see [15]).

3 Logarithmic limit sets of definable sets

3.1 Some properties of definable sets. Let \mathfrak{A} be an o-minimal and polynomially bounded expansion of $\overline{\mathbb{R}}$.

Lemma 3.1. *For every definable function $f : (\mathbb{R}_{>0})^n \rightarrow \mathbb{R}_{>0}$, there is an exponential basic cone C and $N \in \mathbb{N}$ such that $f|_C(x_1, \dots, x_n) \geq (x_n)^N$.*

Proof. Fix an exponential basic cone $C \subset (\mathbb{R}_{>0})^n$. By the Łojasiewicz inequality (see [21, 4.14]) there exist $N \in \mathbb{N}$ and $Q > 0$ such that $Qf|_C(x_1, \dots, x_n) \geq (x_n)^N$. The assertion follows by choosing an exponent bigger than N and a suitable exponential basic cone smaller than C . \square

Lemma 3.2. *Every cell decomposition of $(\mathbb{R}_{>0})^n$ has a cell containing an exponential basic cone.*

Proof. This proof is based on the cell decomposition theorem, see [19, Chapter 3] for details. We proceed by induction on n . For $n = 1$, the statement is trivial.

Suppose the lemma true for n . If $\{C_i\}$ is a cell decomposition of $(\mathbb{R}_{>0})^{n+1}$, and if $\pi : (\mathbb{R}_{>0})^{n+1} \rightarrow (\mathbb{R}_{>0})^n$ is the projection on the first n coordinates, then $\{\pi(C_i)\}$ is a

cell decomposition of $(\mathbb{R}_{>0})^n$, hence, by induction, it contains an exponential basic cone D of $(\mathbb{R}_{>0})^n$. Then $\pi^{-1}(D) \times (0, 1]$ contains a cell of the form

$$E = \{(\bar{x}, x_{n+1}) \mid \bar{x} \in D, 0 < x_{n+1} < f(\bar{x})\},$$

where $\bar{x} = (x_1, \dots, x_n)$ and $f : D \rightarrow (0, 1]$ is definable. By the previous lemma, there is an exponential basic cone $D' \subset D$ and $N \in \mathbb{N}$ such that $f_{|D'}(\bar{x}) \geq (x_n)^N$. Hence E contains the exponential basic cone

$$\{(\bar{x}, x_{n+1}) \mid \bar{x} \in D', 0 < x_{n+1} \leq (x_n)^N\}. \quad \square$$

Corollary 3.3. *Let $V \subset (\mathbb{R}_{>0})^n$ be definable in \mathfrak{R} , and suppose that V contains a sequence $(x(k))$ in standard position in dimension n . Then V contains an exponential basic cone.*

Proof. Let $\{C_i\}$ be a cell decomposition of $(\mathbb{R}_{>0})^n$ adapted to V . By the previous lemma, one of the cells contains an exponential basic cone D . By Lemma 2.6, if k is sufficiently large, $(x(k)) \in D$, hence $D \subset V$. \square

Corollary 3.4. *Let $V \subset (\mathbb{R}_{>0})^2$ be definable in \mathfrak{R} , and suppose that exists a sequence $(x(k))$ in V such that $(x(k)) \rightarrow 0$ and*

$$\forall N \in \mathbb{N} : \exists k_0 : \forall k > k_0 : x_2(k) < (x_1(k))^N.$$

Then there exist $h_0 > 0$ and $M \in \mathbb{N}$ such that

$$\{x \in \mathbb{R}^2 \mid 0 < x_1 < h_0 \text{ and } 0 < x_2 < (x_1)^M\} \subset V.$$

Proof. This is precisely the previous corollary with $n = 2$. \square

Lemma 3.5. *Let $V \subset (\mathbb{R}_{>0})^n$ be definable in \mathfrak{R} , and suppose that there exists a sequence $(x(k))$ in V , an integer $m \in \{1, \dots, n\}$ and, for $g = n - m$, positive numbers $a_1, \dots, a_g > 0$ such that $(x(k)) \rightarrow (a_1, \dots, a_g, 0, \dots, 0)$, and such that*

$$\forall N \in \mathbb{N} : \exists k_0 : \forall k > k_0 : \forall i \in \{g + 1, \dots, n - 1\} : x_n(k) < (x_i(k))^N.$$

Then for every $\varepsilon > 0$ there exist a sequence $(y(k))$ in V and positive real numbers $b_1, \dots, b_{n-1} > 0$ such that $(y(k)) \rightarrow (b_1, \dots, b_{n-1}, 0)$ and for all $i \in \{1, \dots, g\}$ we have $|b_i - a_i| < \varepsilon$.

Proof. If $n = 2$ the statement follows by Corollary 3.4. By induction on n we suppose the statement to be true for definable sets in $\mathbb{R}^{n'}$ with $n' < n$. We split the proof in two cases, when $m < n$ and when $m = n$.

If $m < n$, fix an $\varepsilon > 0$, smaller than every a_i , and consider the parallelepiped

$$c_\varepsilon = \left\{ (z_1, \dots, z_n) \in \mathbb{R}^n \mid |z_1 - a_1| < \frac{1}{2}\varepsilon, \dots, |z_g - a_g| < \frac{1}{2}\varepsilon \right\}.$$

Let $\pi : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be the projection on the last m coordinates. The set $\pi(V \cap c_\varepsilon)$ is definable in \mathbb{R}^m , the sequence $(\pi(x(k)))$ satisfies the hypotheses of the lemma, hence, by induction, there exists a sequence $(z(k)) \in \pi(V \cap c_\varepsilon)$ converging to the point $(b_{g+1}, \dots, b_{n-1}, 0)$. Let $(y(k))$ be a sequence such that $y(k) \in \pi^{-1}(z(k))$. We can extract a subsequence (called again $(y(k))$) such that $(y(k)) \rightarrow (b_1, \dots, b_{n-1}, 0)$ where for all $i \in \{1, \dots, g\}$ we have $|b_i - a_i| \leq \frac{1}{2}\varepsilon$.

If $m = n > 2$, then $(x(k)) \rightarrow 0$. The sequence $(\frac{(x_1(k), \dots, x_{n-1}(k))}{|(x_1(k), \dots, x_{n-1}(k))|})$ is contained in the unit sphere S^{n-2} , and, up to subsequences, we can suppose that it converges to a unit vector $v = (v_1, \dots, v_{n-1}) \in (\mathbb{R}_{\geq 0})^{n-1}$. Up to reordering, $v = (v_1, \dots, v_h, 0, \dots, 0)$, with $v_1, \dots, v_h > 0$. We can choose an $\alpha > 0$ small enough such that the closure of the open cone

$$C_v(\alpha) = \left\{ y \in \mathbb{R}^n \mid \frac{\langle (y_1, \dots, y_{n-1}), v \rangle}{|(y_1, \dots, y_{n-1})||v|} > \cos \alpha \right\}$$

intersects the coordinate hyperplane $\{y \mid y_1 = 0\}$ only in 0. Let $\pi : \mathbb{R}^n \rightarrow \mathbb{R}^{n-1}$ the projection on the last $n-1$ coordinates. The set $\pi(V \cap C_v(\alpha))$ is definable in \mathbb{R}^{n-1} , the sequence $(\pi(x(k)))$ satisfies the hypotheses of the lemma, hence, by induction, there exists a sequence $(z(k))$ in $\pi(V \cap C_v(\alpha))$ converging to the point $(b_2, \dots, b_{n-1}, 0)$, with $b_2, \dots, b_{n-1} > 0$. Let $(y(k))$ be a sequence such that $y(k) \in \pi^{-1}(z(k))$. Up to subsequences, $(y(k)) \rightarrow (b_1, \dots, b_{n-1}, 0)$. To see that $b_1 > 0$, note that $y(k) \in C_v(\alpha)$, whose closure intersects $\{y \mid y_1 = 0\}$ only in 0. Hence if, by contradiction, we have that $(y_1(k)) \rightarrow 0$, then $(y(k)) \rightarrow 0$. But $b_2, \dots, b_{n-1} > 0$, a contradiction. \square

3.2 Polyhedral structure. Let $V \subset (\mathbb{R}_{>0})^n$ be a definable set in an o-minimal and polynomially bounded structure. Our main object of study is $\mathcal{A}_0(V)$, the logarithmic limit set of V . Suppose that \mathfrak{R} has field of exponents $\Lambda \subset \mathbb{R}$. Given a matrix $B \in \text{GL}_n(\Lambda)$, the set $B(\mathcal{A}_0(V))$ is the logarithmic limit set of $\overline{B}(V)$. The components of \overline{B}^{-1} are all definable in \mathfrak{R} because their exponents are in Λ , hence the set $\overline{B}(V)$ is again definable.

Theorem 3.6. *Let $V \subset (\mathbb{R}_{>0})^n$ be a set definable in an o-minimal and polynomially bounded structure. The point $(0, \dots, 0, -1)$ is in $\mathcal{A}_0(V)$ if and only if there exists a sequence $(x(k))$ in V such that $(x(k)) \rightarrow (a_1, \dots, a_{n-1}, 0)$, where $a_1, \dots, a_{n-1} > 0$.*

Proof. If there exists such an $(x(k))$, then it is obvious that $(0, \dots, 0, -1) \in \mathcal{A}_0(V)$. Vice versa, if $(0, \dots, 0, -1) \in \mathcal{A}_0(V)$, then by Lemma 2.3 there exists a sequence $(y(k))$ in V such that $(y_n(k)) \rightarrow 0$ and

$$\forall N \in \mathbb{N} : \exists k_0 \in \mathbb{N} : \forall k > k_0 : \forall i \in \{1, \dots, n-1\} : \\ y_n(k) < (y_i(k))^N \text{ and } y_n(k) < (y_i(k))^{-N}.$$

Up to subsequences we can suppose that for all $i \in \{1, \dots, n-1\}$ one of the following three possibilities occur: $(y_i(k)) \rightarrow 0$, $(y_i(k)) \rightarrow a_i \neq 0$, $(y_i(k)) \rightarrow +\infty$. Up to a change of coordinates with maps of the form $B_i(x_1, \dots, x_i, \dots, x_n) = (x_1, \dots, -x_i, \dots, x_n)$, we can suppose that for all $i \in \{1, \dots, n-1\}$ either $(y_i(k)) \rightarrow 0$ or $(y_i(k)) \rightarrow a_i \neq 0$. Then we can apply Lemma 3.5, and we are done. \square

Now we suppose that \mathfrak{A} is a regular polynomially bounded structure, or, equivalently, that \mathfrak{A} has field of exponents \mathbb{R} . Let $x \in \mathcal{A}_0(V)$. We want to describe a neighborhood of x in $\mathcal{A}_0(V)$. To do this, we choose a map $B \in \text{GL}_n(\mathbb{R})$ such that $B(x) = (0, \dots, 0, -1)$. Now we only need to describe a neighborhood of $(0, \dots, 0, -1)$ in $\mathcal{A}_0(\overline{B}(V))$. As logarithmic limit sets are cones, we only need to describe a neighborhood of 0 in

$$H = \{(x_1, \dots, x_{n-1}) \in \mathbb{R}^{n-1} \mid (x_1, \dots, x_{n-1}, -1) \in \mathcal{A}_0(\overline{B}(V))\}.$$

We define a one-parameter family and its limit (in the sense of subsection 2.1):

$$\begin{aligned} W_t &= \{(x_1, \dots, x_{n-1}) \in (\mathbb{R}_{>0})^{n-1} \mid (x_1, \dots, x_{n-1}, t) \in \overline{B}(V)\}, \\ W &= \left(\lim_{t \rightarrow 0} W_t\right) \cap (\mathbb{R}_{>0})^{n-1}. \end{aligned}$$

The set W is a definable subset of $(\mathbb{R}_{>0})^{n-1}$. Its logarithmic limit set is denoted, as usual, by $\mathcal{A}_0(W) \subset \mathbb{R}^{n-1}$. By the previous theorem, as $(0, \dots, 0, -1) \in \mathcal{A}_0(\overline{B}(V))$, W is not empty, hence $0 \in \mathcal{A}_0(W)$. We want to prove that there exists a neighborhood U of 0 in \mathbb{R}^{n-1} such that $\mathcal{A}_0(W) \cap U = H \cap U$ or, in other words, that $\mathcal{A}_0(W) \cap H$ is a neighborhood of 0 both in $\mathcal{A}_0(W)$ and H . This will be achieved in Theorem 3.10.

A *flag* in \mathbb{R}^n is a sequence (V_0, V_1, \dots, V_h) , $h \leq n$, of subspaces of \mathbb{R}^n such that $V_0 \subset V_1 \subset \dots \subset V_h \subset \mathbb{R}^n$ and $\dim V_i = i$. We use flags to encode the direction in which a sequence converges to a point, in the following sense: we say that a sequence $(x(k))$ in \mathbb{R}^n converges to the point y *along* the flag (V_1, V_2, \dots, V_h) if $(x(k)) \rightarrow y$, for $k : x(k) - y \in V_h \setminus V_{h-1}$ and $\forall i \in \{0, \dots, h-2\}$, the sequence $(\pi_i(x(k) - y))$ converges to the point $\pi_i(V_{i+1})$, where $\pi_i : V_h \setminus V_i \rightarrow \mathbb{P}(V_h/V_i)$ is the canonical projection on the projective space $\mathbb{P}(V_h/V_i)$.

Lemma 3.7. *For all sequences $(x(k))$ in \mathbb{R}^n converging to a point y , there exists a flag (V_0, \dots, V_h) and a subsequence of $(x(k))$ converging to y along (V_0, \dots, V_h) .*

Proof. It follows from the compactness of $\mathbb{P}(V_h/V_i)$. □

Lemma 3.8. *Let $x(k) \subset H$ be a sequence converging to 0. Then at least one of its points is in $\mathcal{A}_0(W) \cap H$. In particular $\mathcal{A}_0(W) \cap H$ contains a neighborhood of 0 in H .*

Proof. By Lemma 3.7, we can extract a subsequence, again denoted by $x(k)$, converging to zero along a flag (V_0, V_1, \dots, V_h) in \mathbb{R}^{n-1} . Up to a linear change of coordinates, we can suppose that this flag is given by $(\{0\}, \text{Span}(e_{n-1}), \text{Span}(e_{n-2}, e_{n-1}), \dots, \text{Span}(e_{n-h}, \dots, e_{n-1}))$. Hence for $i \in \{1, \dots, n-h-1\}$ we have $x_i(k) = 0$. Again by extracting a subsequence and by a change of coordinates with maps of the form

$$B_i(x_1, \dots, x_i, \dots, x_n) = (x_1, \dots, -x_i, \dots, x_n),$$

with $i \in \{n-h, \dots, n-1\}$, we can suppose that for all such i , $x_i(k) < 0$.

By Proposition 2.2, as $H \times \{-1\} \subset \mathcal{A}_0(\overline{B}(V))$, for every point $x(k)$ there exists a sequence $(y(k, l))$ in $\overline{B}(V)$ and a sequence $(t(k, l))$ in $(0, 1)$ such that $(t(k, l)) \rightarrow 0$

and $(\text{Log}_{\frac{1}{\tau(k,l)}}(y(k,l))) \rightarrow (x(k), -1)$. By Theorem 3.6 we can choose $(y(k,l))$ such that $(y(k,l)) \rightarrow a(k)$, with $a_i(k) > 0$ for $i \in \{1, \dots, n-h-1\}$, and $a_i(k) = 0$ for $i \in \{n-h, \dots, n\}$. Up to a change of coordinates with maps of the form

$$B_i(x_1, \dots, x_i, \dots, x_n) = (x_1, \dots, -x_i, \dots, x_n),$$

with $i \in \{1, \dots, n-h-1\}$, we can suppose that the sequence $a(k)$ is bounded and that, up to subsequences, it converges to a point a , with $a_i = 0$ for $i \in \{n-h, \dots, n\}$.

Let $\pi : \mathbb{R}^n \rightarrow \mathbb{R}^{n-h-1}$ be the projection on the first $n-h-1$ coordinates. Then $\pi(a(k)) \subset (\mathbb{R}_{>0})^{n-h-1}$. By Lemma 2.5 we can suppose that $\pi(a(k))$ is in standard position, i.e. $a_1, \dots, a_g > 0, a_{g+1} = \dots = a_n = 0$ and

$$\forall N \in \mathbb{N} : \exists k_0 : \forall k > k_0 : \forall i \in \{g+1, \dots, n-h-2\} : a_{i+1}(k) < (a_i(k))^N.$$

From the sequences $y(k,l)$, we extract a diagonal subsequence $z(k)$ in the following way. For every k , the sequence $y(k,l)$ converges to $a(k) = (a_1(k), \dots, a_{n-h-1}(k), 0, \dots, 0)$. As $\text{Log}_{\frac{1}{\tau(k,l)}}(y(k,l)) \rightarrow (x(k), -1) = (0, \dots, 0, x_{n-h}(k), \dots, x_{n-1}(k), -1)$, for all $i \in \{n-h, \dots, n-1\}$ we have

$$\frac{\log_e(y_n(k,l))}{\log_e(y_i(k,l))} \rightarrow \frac{-1}{x_i(k)}.$$

For every k , we can choose an $l_0(k)$ such that

- (1) $\forall i \in \{n-h, \dots, n-1\} : \left| \frac{\log_e(y_n(k, l_0(k)))}{\log_e(y_i(k, l_0(k)))} - \frac{-1}{x_i(k)} \right| < \frac{1}{k}$,
- (2) $|y(k, l_0(k)) - a(k)| < \frac{1}{k}$.

We define $z(k) = y(k, l_0(k))$. Now $(z(k)) \rightarrow a = (a_1, \dots, a_g, 0, \dots, 0)$ and, as $x(k) \rightarrow (0, \dots, 0)$ along the flag (V_0, \dots, V_h) , we have

$$\forall N \in \mathbb{N} : \exists k_0 : \forall k > k_0 : \forall i \in \{g+1, \dots, n-1\} : z_{i+1}(k) < (z_i(k))^N.$$

Let r be smaller than every a_1, \dots, a_g . Consider the parallelepiped

$$c_r = \left\{ (z_1, \dots, z_n) \in \mathbb{R}^n \mid |z_1 - a_1| < \frac{1}{2}r, \dots, |z_g - a_g| < \frac{1}{2}r \right\}.$$

Let $\pi : \mathbb{R}^n \rightarrow \mathbb{R}^{n-g}$ be the projection on the last $n-g$ coordinates. The set $\pi(\overline{B}(V) \cap c_r)$ is definable in \mathbb{R}^{n-g} , and the sequence $\pi(z(k))$ satisfies the hypotheses of Corollary 3.3, hence $\pi(\overline{B}(V) \cap c_r)$ contains an exponential basic cone, hence $\pi(W \times \{-1\} \cap c_r)$ also contains one. This means that $\mathcal{A}_0(\pi(W \times \{-1\}) \cap c_r)$ contains a basic cone. Hence also $\mathcal{A}_0(W \times \{-1\} \cap c_r)$ contains this cone, and also $\mathcal{A}_0(W)$. By Lemma 2.6, at least one of the points $z(k)$ is in this cone. \square

Lemma 3.9. *Let $x \in \mathcal{A}_0(W)$. Then the number $r(x) = \sup\{r \mid \lambda x \in H, \text{ if } 0 \leq \lambda \leq r\}$ is strictly positive.*

Proof. Let $x \in \mathcal{A}_0(W)$. By a linear change of coordinates, we can suppose that $x = (0, \dots, 0, -1) \in \mathbb{R}^{n-1}$. By Theorem 3.6 there is a sequence $(x(k))$ in W converging to the point $(a_1, \dots, a_{n-2}, 0)$, with $a_1, \dots, a_{n-2} > 0$. As W is the limit of the family W_t , for every k there is a sequence $(y(k, l))$ in $\overline{B}(V)$ converging to $(x(k), 0)$. We can construct a diagonal sequence $(z(k))$ in $\overline{B}(V)$ in the following way: for every k we can choose an $l_0(k)$ such that

$$|y(k, l_0(k)) - (x(k), 0)| < (x_{n-1}(k))^k.$$

Then we set $z(k) = y(k, l_0(k))$. The sequence $z(k)$ converges to the point $(a_1, \dots, a_{n-2}, 0, 0)$. Let r be smaller than any of the a_1, \dots, a_{n-2} . Consider the parallelepiped

$$c_r = \left\{ (z_1, \dots, z_n) \in \mathbb{R}^n \mid |z_1 - a_1| < \frac{1}{2}r, \dots, |z_{n-2} - a_{n-2}| < \frac{1}{2}r \right\}.$$

Let $\pi : \mathbb{R}^n \rightarrow \mathbb{R}^2$ be the projection on the last 2 coordinates. The set $\pi(\overline{B}(V) \cap c_r)$ is a definable subset of \mathbb{R}^2 , and the sequence $\pi(z(k))$ satisfies the hypotheses of Corollary 3.4, hence $\pi(\overline{B}(V) \cap c_r)$ contains an exponential basic cone. This means that there exists a number $r' > 0$ such that $\{(0, \dots, 0, z) \mid -r' \leq z \leq 0\} \subset H$. \square

Theorem 3.10. *Let $V \subset (\mathbb{R}_{>0})^n$ be a set definable in a regular polynomially bounded structure. Let $x \in \mathcal{A}_0(V)$ and choose a map $B \in \text{GL}_n(\mathbb{R})$ such that $B(x) = (0, \dots, 0, -1)$. We recall that*

$$\begin{aligned} H &= \{(x_1, \dots, x_{n-1}) \in \mathbb{R}^{n-1} \mid (x_1, \dots, x_{n-1}, -1) \in \mathcal{A}_0(\overline{B}(V))\} \\ W_t &= \{(x_1, \dots, x_{n-1}) \in (\mathbb{R}_{>0})^{n-1} \mid (x_1, \dots, x_{n-1}, t) \in \overline{B}(V)\} \\ W &= \left(\lim_{t \rightarrow 0} W_t \right) \cap (\mathbb{R}_{>0})^{n-1}. \end{aligned}$$

Then there exists a neighborhood U of 0 in \mathbb{R}^{n-1} such that $\mathcal{A}_0(W) \cap U = H \cap U$.

Proof. We will prove that $\mathcal{A}_0(W) \cap H$ is a neighborhood of 0 both in $\mathcal{A}_0(W)$ and in H . The previous lemma implies that if $(x(k))$ is a sequence in H converging to 0, then at least one of its points is in $\mathcal{A}_0(W)$, hence $\mathcal{A}_0(W) \cap H$ is a neighborhood of 0 in H .

To prove that $\mathcal{A}_0(W) \cap H$ is also a neighborhood of 0 in $\mathcal{A}_0(W)$, we only need to prove that if r is the function defined in Lemma 3.9, there exists an $\varepsilon > 0$ such that

$$\forall x \in \mathcal{A}_0(W) \cap S^{m-2} : r(x) > \varepsilon.$$

But this is true, because we already know that $\mathcal{A}_0(W) \cap H$ is a neighborhood of 0 in H . \square

Theorem 3.11. *Let $V \subset (\mathbb{R}_{>0})^n$ be a set definable in a regular polynomially bounded structure. The logarithmic limit set $\mathcal{A}_0(V)$ is a polyhedral complex. Moreover, if $\dim V = m$, then $\dim \mathcal{A}_0(V) \leq m$.*

Proof. By induction on n . For $n = 1$ the statement is trivial, as a cone in \mathbb{R} is a polyhedral set, and every zero dimensional definable set is compact, hence its logarithmic limit set is a point. Suppose the statement is true for $n - 1$. For every $x \in \mathcal{A}_0(V)$ there is a linear map B sending x to $(0, \dots, 0, -1)$. The statement in [21, 4.7] implies that the definable set $W \subset (\mathbb{R}_{>0})^n$ has dimension less than or equal to $m - 1$, hence $\mathcal{A}_0(W)$ is a polyhedral set of dimension less than or equal to $m - 1$ (by the inductive hypothesis). By the previous theorem a neighborhood of the ray $\{\lambda x \mid \lambda \geq 0\}$ in $\mathcal{A}_0(V)$ is the cone over a neighborhood of 0 in $\mathcal{A}_0(W)$, hence it is a polyhedral complex of dimension less than or equal to m . By compactness of the sphere S^{m-1} , $\mathcal{A}_0(V)$ can be covered by a finite number of such neighborhoods, hence it is a polyhedral complex of dimension less than or equal to m . \square

Note that the statement about the dimension can be false for a general set. See Figure 4 for an example.

Moreover, it is not possible to give more than an inequality, as for every $s \leq m$ it is always possible to find a semi-algebraic set $V \subset (\mathbb{R}_{>0})^m$ such that $\dim V = m$ and $\dim \mathcal{A}_0(V) = s$. For example take the parallelepiped $V = [1, 2]^{m-s} \times (\mathbb{R}_{>0})^s \subset (\mathbb{R}_{>0})^m$, with $\mathcal{A}_0(V) = \{0\}^{m-s} \times (\mathbb{R}_{>0})^s$. It is also possible to find counterexamples of this kind where V is the intersection of $(\mathbb{R}_{>0})^{m+1}$ with an algebraic hypersurface. For example let $S^{m-s} \subset (\mathbb{R}_{>0})^{m-s+1}$ be the sphere with center $(2, \dots, 2)$ and radius 1, then $V = S^{m-s} \times (\mathbb{R}_{>0})^s \subset (\mathbb{R}_{>0})^{m+1}$ has dimension m , but $\mathcal{A}_0(V) = \{0\}^{m-s+1} \times (\mathbb{R}_{>0})^s$ has dimension s .

It is also possible to find a semi-algebraic set V that is the intersection of $(\mathbb{R}_{>0})^n$ with an irreducible pure-dimensional smooth hypersurface, and such that its logarithmic limit set $\mathcal{A}_0(V)$ is not pure-dimensional, see for example Figure 5. Note that the product $V \times S^h$, with S^h the sphere with center $(2, \dots, 2)$ and radius 1 as above, is again the intersection of $(\mathbb{R}_{>0})^{n+h+1}$ with an irreducible pure-dimensional smooth variety, and its logarithmic limit set is lower dimensional and not pure-dimensional.

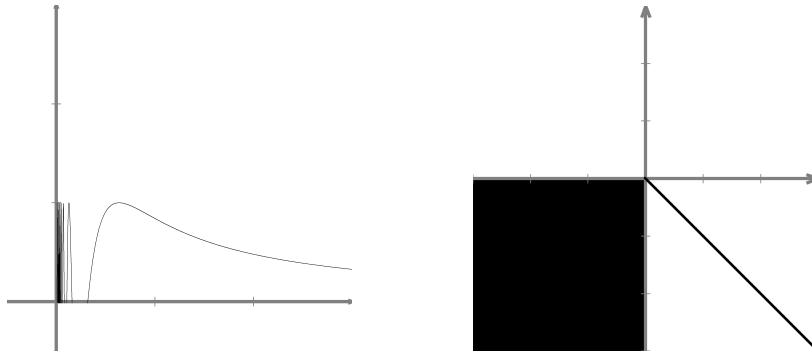


Figure 4. If $V = \{(x, y) \in (\mathbb{R}_{>0})^2 \mid y = \sin \frac{1}{x}\}$ (left picture), then $\mathcal{A}_0(V) = \{(x, y) \in \mathbb{R}^2 \mid y \leq 0, x \leq 0 \text{ or } x \geq 0, y = -x\}$ (right picture).

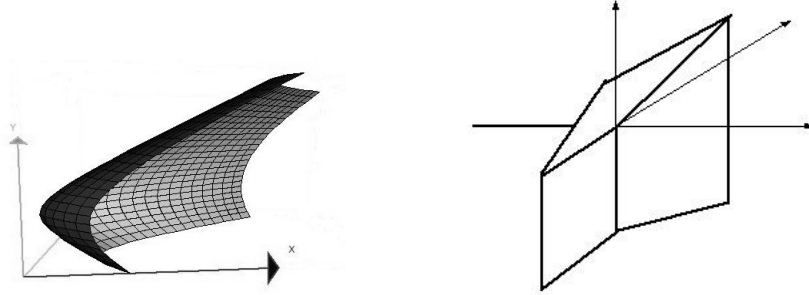


Figure 5. If $V = \{(x, y, z) \in (\mathbb{R}_{>0})^3 \mid x = (y-1)^2 + (z-1)^2\}$ (left picture), then $\mathcal{A}_0(V)$ contains a ray along the direction $(-1, 0, 0)$ that is not contained in any two-dimensional face, and a two-dimensional part made up of three infinite two-dimensional faces in the half-space $x \geq 0$ (right picture).

4 Non-Archimedean description

4.1 The Hardy field. Let S be a set of symbols expanding \mathcal{OS} , and let $\mathfrak{R} = (\mathbb{R}, a)$ be an o-minimal S -structure expanding $\overline{\mathbb{R}}$ (see subsection 2.3 for definitions).

The *Hardy field* of \mathfrak{R} can be defined in the following way. If $f, g : \mathbb{R}_{>0} \rightarrow \mathbb{R}$ are two definable functions, we say that they have the same *germ* near zero, and we write $f \sim g$, if there exists an $\varepsilon > 0$ such that $f|_{(0,\varepsilon)} = g|_{(0,\varepsilon)}$. The Hardy field can be defined as the set of germs of definable functions near zero: $H(\mathfrak{R}) = \{f : \mathbb{R}_{>0} \rightarrow \mathbb{R} \mid f \text{ definable}\} / \sim$. We will denote by $[f]$ the germ of a function f .

For every element $a \in \mathbb{R}$, the constant function with value a defines a germ that is identified with a . This defines an embedding $\mathbb{R} \rightarrow H(\mathfrak{R})$. Every relation in the structure \mathfrak{R} defines a corresponding relation on $H(\mathfrak{R})$, and every function in the structure \mathfrak{R} defines a function on $H(\mathfrak{R})$, hence the Hardy field $H(\mathfrak{R})$ can be endowed with an S -structure $\overline{H(\mathfrak{R})}$. Given an L_S -formula $\varphi(x_1, \dots, x_n)$, and given definable functions f_1, \dots, f_n , we have

$$\overline{H(\mathfrak{R})} \models \varphi([f_1], \dots, [f_n]) \iff \exists \varepsilon > 0 : \forall t \in (0, \varepsilon) : \mathfrak{R} \models \varphi(f_1(t), \dots, f_n(t)).$$

See [7, Section 5.3] for precise definitions and proofs. In particular the S -structure $\overline{H(\mathfrak{R})}$ is an elementary extension of the S -structure \mathfrak{R} . Note that the operations $+$ and \cdot turn $H(\mathfrak{R})$ into a field, the order \leq turns it into an ordered field, and also note that this field is real closed. Moreover, the S -structure $\overline{H(\mathfrak{R})}$ is o-minimal.

Suppose that S' is an expansion of S , and that \mathfrak{R}' is an S' -structure expanding \mathfrak{R} . Then all functions that are definable in \mathfrak{R} are also definable in \mathfrak{R}' . This defines an inclusion $H(\mathfrak{R}) \subset H(\mathfrak{R}')$. Note that, by restriction, \mathfrak{R}' has an S -structure induced by his S' -structure. If $\varphi(x_1, \dots, x_n)$ is an L_S -formula, and $h_1, \dots, h_n \in H(\mathfrak{R})$, then

$$\overline{H(\mathfrak{R})} \models \varphi(h_1, \dots, h_n) \iff \overline{H(\mathfrak{R}')} \models \varphi(h_1, \dots, h_n).$$

In other words the S -structure on $H(\mathfrak{R})$ is an elementary extension of $\overline{H(\mathfrak{R})}$.

If \mathfrak{R} is polynomially bounded, for every definable function f whose germ is not 0, there exists r in the field of exponents and $c \in \mathbb{R} \setminus \{0\}$ such that

$$\lim_{x \rightarrow 0^+} \frac{f(x)}{x^r} = c.$$

If h is the germ of f , we denote the exponent r by $v(h)$. The map $v : H(\mathfrak{R}) \setminus \{0\} \rightarrow \mathbb{R}$ is a real valued *valuation*, turning $H(\mathfrak{R})$ in a *non-Archimedean field* of rank one. The image group of the valuation is the field of exponents of \mathfrak{R} , denoted by Λ . The valuation has a natural section, the map

$$\Lambda \ni r \rightarrow x^r \in H(\mathfrak{R}).$$

The *valuation ring*, denoted by \mathcal{O} , is the set of all germs bounded in a neighborhood of zero, and the *maximal ideal* m of \mathcal{O} is the set of all germs infinitesimal in zero. The valuation ring \mathcal{O} is convex with respect to the order \leq , hence the valuation topology coincides with the order topology. The map $\mathcal{O} \rightarrow \mathbb{R}$ sending every element of \mathcal{O} to its value in zero has kernel m , hence it identifies in a natural way the residue field \mathcal{O}/m with \mathbb{R} . We will usually denote by $t \in H(\mathfrak{R})$ the germ of the identity function. We have $v(t) = 1$.

As an example we can describe the field $H(\overline{\mathbb{R}})$. Every element of this field is algebraic over the fraction field $\mathbb{R}(t)$. Hence $H(\overline{\mathbb{R}})$ is the real closure of $\mathbb{R}(t)$, with reference to the unique order such that $t > 0$ and $\forall x \in \mathbb{R}_{>0} : t < x$. The image of the valuation is \mathbb{Q} . Consider the real closed field of formal Puiseux series with real coefficients, $\mathbb{R}((t^{\mathbb{Q}})) = \bigcup_{n \geq 1} \mathbb{R}((t^{1/n}))$. The elements of this field have the form $x^r(s(x^{1/n}))$, where $r \in \mathbb{Z}$ and s is a formal power series. We have $\mathbb{R}(t) \subset \mathbb{R}((t^{\mathbb{Q}}))$ as ordered fields, hence $H(\overline{\mathbb{R}}) \subset \mathbb{R}((t^{\mathbb{Q}}))$. The elements of $H(\overline{\mathbb{R}})$ are the elements of $\mathbb{R}((t^{\mathbb{Q}}))$ that are algebraic over $\mathbb{R}(t)$. For these elements the formal power series s is locally convergent.

Another example is the field $H(\mathbb{R}_{\text{an}}^*)$, see [22]. By [22, Theorem B], for every element h of this field, there exists a number $r \in \mathbb{R}$, a formal power series

$$F = \sum_{\alpha \in \mathbb{R}_{\geq 0}} c_{\alpha} X^{\alpha}$$

and a radius $\delta \in \mathbb{R}_{>0}$ such that $c_{\alpha} \in \mathbb{R}$, $\{\alpha \mid c_{\alpha} \neq 0\}$ is well ordered, the series $\sum_{\alpha} |c_{\alpha}| \delta^{\alpha} < +\infty$, (hence F is convergent and defines a continuous function on $[0, \delta]$, analytic on $(0, \delta)$) and

$$h = [x^r F(x)].$$

Let \mathbb{F} be a real closed field extending \mathbb{R} . The convex hull of \mathbb{R} in \mathbb{F} is a valuation ring denoted by \mathcal{O}_{\leq} . This valuation ring defines a valuation $v : \mathbb{F}^* \rightarrow \Lambda$, where Λ is an ordered abelian group. We say that \mathbb{F} is a real closed *non-Archimedean field* of rank one extending \mathbb{R} if Λ has rank one as an ordered group, or, equivalently, if Λ is isomorphic to an additive subgroup of \mathbb{R} . Hence real closed non-Archimedean fields of rank one extending \mathbb{R} have a real valued valuation (non necessarily surjective) well defined up to

a scaling factor. This valuation is well defined when we choose an element $t \in \mathbb{F}$ with $t > 0$ and $v(t) > 0$, and we choose a scaling factor such that $v(t) = 1$. Now a valuation $v : \mathbb{F} \rightarrow \mathbb{R}$ is well defined, with image $v(\mathbb{F}^*) = \Lambda \subset \mathbb{R}$.

Consider the subfield $\mathbb{R}(t) \subset \mathbb{F}$. The order induced by \mathbb{F} has the property that $t > 0$ and $\forall x \in \mathbb{R}_{>0} : t < x$. Hence \mathbb{F} contains the real closure of $\mathbb{R}(t)$ with reference to this order, i.e. $H(\overline{\mathbb{R}})$. Moreover the valuation v on \mathbb{F} restricts to the valuation we have defined on $H(\overline{\mathbb{R}})$, as, if \mathcal{O}_{\leq} is the valuation ring of \mathbb{F} , $\mathcal{O}_{\leq} \cap H(\overline{\mathbb{R}})$ is precisely the valuation ring \mathcal{O} of $H(\overline{\mathbb{R}})$. In other words every non-Archimedean real closed field \mathbb{F} of rank one extending \mathbb{R} is a valued extension of $H(\overline{\mathbb{R}})$.

4.2 Non-Archimedean amoebas. Let \mathbb{F} be a non-Archimedean real closed field of rank one extending \mathbb{R} , with a fixed real valued valuation $v : \mathbb{F}^* \rightarrow \mathbb{R}$. By convention, we define $v(0) = \infty$, an element greater than any element of \mathbb{R} . The map

$$\mathbb{F} \ni h \rightarrow \|h\| = \exp(-v(h)) \in \mathbb{R}_{\geq 0}$$

is a *non-Archimedean norm*. The *component-wise logarithm map* can be defined also on \mathbb{F} by

$$\text{Log} : (\mathbb{F}_{>0})^n \ni (h_1, \dots, h_n) \rightarrow (\log(\|h_1\|), \dots, \log(\|h_n\|)) \in \mathbb{R}^n.$$

Note that $\log(\|h\|) = -v(h)$. If $V \subset (\mathbb{F}_{>0})^n$, the *logarithmic image* of V is the image $\text{Log}(V)$.

Let S be a set of symbols expanding \mathcal{OS} , and let (\mathbb{F}, a) be an S -structure expanding the \mathcal{OS} -structure on the non-Archimedean real closed field \mathbb{F} of rank one extending \mathbb{R} . If $V \subset (\mathbb{F}_{>0})^n$ is a definable set in (\mathbb{F}, a) , we define the *non-Archimedean amoeba* of V as the closure in \mathbb{R}^n of the logarithmic image of V , and we will write $\mathcal{A}(V) = \overline{\text{Log}(V)}$.

The case we are more interested in is when $\mathfrak{R} = (\mathbb{R}, a)$ is an o -minimal, polynomially bounded S -structure expanding $\overline{\mathbb{R}}$, and $H(\mathfrak{R})$ is the Hardy field, with its natural valuation v and its natural S -structure. Non-Archimedean amoebas of definable sets of $H(\mathfrak{R})$ are closely related with logarithmic limit sets of definable sets of \mathfrak{R} .

Let $\mathbb{F} \subset \mathbb{K}$ be two real closed fields. Let S be a set of symbols expanding \mathcal{OS} , let $(\mathbb{F}, a), (\mathbb{K}, b)$ be S -structures expanding the \mathcal{OS} structure on the real closed fields and such that \mathbb{K} is an elementary extension of \mathbb{F} . Let $V \subset \mathbb{F}^n$ be a definable set in (\mathbb{F}, a) . We will denote by $V^{\mathbb{K}}$ the *extension* of V to the structure (\mathbb{K}, b) .

For example, if $V \subset \mathbb{R}^n$ is a definable set in \mathfrak{R} , we can always define an extension $V^{H(\mathfrak{R})} \subset H(\mathfrak{R})^n$ of V to $H(\mathfrak{R})$.

Lemma 4.1. *Let \mathfrak{R} be an o -minimal polynomially bounded structure. Let $V \subset (\mathbb{R}_{>0})^n$ be a definable set. Then $(0, \dots, 0, -1) \in \mathcal{A}_0(V) \iff (0, \dots, 0, -1) \in \text{Log}(V^{H(\mathfrak{R})})$.*

Proof. Suppose that $(0, \dots, 0, -1) \in \mathcal{A}(V^{H(\mathfrak{R})})$. Then there is a point $(x_1, \dots, x_n) \in V^{H(\mathfrak{R})}$ such that $v(x_n) = 1$ and $v(x_i) = 0$ for all $i < n$. If f_1, \dots, f_n are definable functions such that $x_i = [f_i]$, then $\exists \varepsilon > 0 : \forall t \in (0, \varepsilon) : (f_1(t), \dots, f_n(t)) \in V$. Moreover, when $t \rightarrow 0$ we have that $f_n(t) \rightarrow 0$ and $f_i(t) \rightarrow a_i > 0$ for $i < n$. Hence

V contains a sequence tending to $(a_1, \dots, a_{n-1}, 0)$ with $a_1, \dots, a_{n-1} \neq 0$, and $\mathcal{A}_0(V)$ contains $(0, \dots, 0, -1)$.

Vice versa, suppose that $(0, \dots, 0, -1) \in \mathcal{A}_0(V)$. Then, by Theorem 3.6 there exists a sequence $(x(k))$ in V such that $(x(k)) \rightarrow (a_1, \dots, a_{n-1}, 0)$, where $a_1, \dots, a_{n-1} > 0$. Let ε be a number less than all the numbers a_1, \dots, a_{n-1} , and consider the set

$$\{x \in \mathbb{R} \mid \exists x_1, \dots, x_{n-1} : |x_i - a_i| < \frac{1}{2}\varepsilon \text{ and } (x_1, \dots, x_{n-1}, x) \in V\}.$$

As this set is definable, and as it contains a sequence converging to zero, it must contain an interval of the form $(0, \delta)$, with $\delta > 0$. In one formula:

$$\forall x \in (0, \delta) : \exists x_1, \dots, x_{n-1} : |x_i - a_i| < \frac{1}{2}\varepsilon \text{ and } (x_1, \dots, x_{n-1}, x) \in V.$$

This sentence can be turned into a first order L_S -formula using a definition of V . This formula must also hold for $H(\mathfrak{R})$. We can choose an $x \in H(\mathfrak{R})$, with $x > 0$ and $v(x) = 1$. Then $x < \delta$, hence

$$\exists x_1, \dots, x_{n-1} : |x_i - a_i| < \frac{1}{2}\varepsilon \text{ and } (x_1, \dots, x_{n-1}, x) \in V^{H(\mathfrak{R})}.$$

Now $v(x_i) = 0$ for all $i > 1$, as $|x_i - a_i| < \frac{1}{2}\varepsilon$. Hence $\text{Log}(x_1, \dots, x_{n-1}, x) = (0, \dots, 0, -1)$. \square

Theorem 4.2. *Let \mathfrak{R} be an o-minimal polynomially bounded structure with field of exponents Λ . Let $V \subset (\mathbb{R}_{>0})^n$ be a definable set. Then*

$$\mathcal{A}_0(V) \cap \Lambda^n = \text{Log}(V^{H(\mathfrak{R})}).$$

In particular, $\text{Log}(V^{H(\mathfrak{R})})$ is a closed subset of Λ^n .

Proof. As $\text{Log}(V^{H(\mathfrak{R})}) \subset \Lambda^n$, we only need to prove that for all $x \in \Lambda^n$, $x \in \mathcal{A}_0(V) \Leftrightarrow x \in \text{Log}(V^{H(\mathfrak{R})})$. We choose a matrix B with entries in Λ sending x to $(0, \dots, 0, -1)$. Then we conclude by the previous lemma applied to the definable set $\overline{B}(V)$. For the last statement, recall that $\mathcal{A}_0(V)$ is a closed subset of \mathbb{R}^n , hence $\mathcal{A}_0(V) \cap \Lambda^n$ is a closed subset of Λ^n . \square

Theorem 4.3. *Let $\mathbb{F} \subset \mathbb{K}$ be two non-Archimedean real closed fields of rank one extending \mathbb{R} , with a choice of a real valued valuation defined by an element $t \in \mathbb{F}$. Denote the value groups by $\Lambda = v(\mathbb{F}^*)$ and $\Omega = v(\mathbb{K}^*)$. Let S be a set of symbols expanding \mathcal{OS} , let $(\mathbb{F}, a), (\mathbb{K}, b)$ be S -structures expanding the \mathcal{OS} structure on the real closed fields and such that \mathbb{K} is an elementary extension of \mathbb{F} . Let V be a definable set in (\mathbb{F}, a) , and $V^{\mathbb{K}}$ be its extension to (\mathbb{K}, b) . Then $\text{Log}(V) \subset \Lambda^n$ is dense in $\text{Log}(V^{\mathbb{K}}) \subset \Omega^n$.*

Proof. Suppose, by contradiction, that $x \in \text{Log}(V^{\mathbb{K}})$ and it is not in the closure of $\text{Log}(V)$. Then there exists an $\varepsilon > 0$ such that the cube

$$C = \{y \in \mathbb{R}^n \mid |y_1 - x_1| < \varepsilon, \dots, |y_n - x_n| < \varepsilon\}$$

does not contain points of $\text{Log}(V)$. Let $h \in V^{\mathbb{K}}$ be an element such that $\text{Log}(h) = x$, and let $d \in \mathbb{F}$ be an element such that $0 < v(d) < \varepsilon$. Consider the cube

$$E = \left(\frac{h_1}{d}, h_1 d\right) \times \left(\frac{h_2}{d}, h_2 d\right) \times \cdots \times \left(\frac{h_n}{d}, h_n d\right) \subset \mathbb{K}^n.$$

The image $\text{Log}(E)$ is contained in C , hence $E \cap V$ is empty. But, as (\mathbb{K}, b) is an elementary extension of (\mathbb{F}, a) , also $E \cap V^{\mathbb{K}}$ is empty. This is a contradiction as $h \in E$ and $h \in V^{\mathbb{K}}$. \square

Corollary 4.4. *Let S be a set of symbols expanding \mathcal{OS} , and let $\mathfrak{R} = (\mathbb{R}, a)$ be an o-minimal polynomially bounded S -structure with field of exponents Λ , expanding $\overline{\mathbb{R}}$. Let $V \subset (\mathbb{R}_{>0})^n$ be a definable set in \mathfrak{R} . Suppose that there exists a subfield $\Omega \subset \mathbb{R}$ such that $\Lambda \subset \Omega$ and \mathfrak{R}^Ω is o-minimal and polynomially bounded. Then $\mathcal{A}_0(V) \cap \Lambda^n$ is dense in $\mathcal{A}_0(V) \cap \Omega^n$.*

Proof. Consider the Hardy fields $H(\mathfrak{R})$ and $H(\mathfrak{R}^\Omega)$. By Theorem 4.2, $\mathcal{A}_0(V) \cap \Lambda^n = \text{Log}(V^{H(\mathfrak{R})})$ and $\mathcal{A}_0(V) \cap \Omega^n = \text{Log}(V^{H(\mathfrak{R}^\Omega)})$. As we said above, the S -structures on $H(\mathfrak{R})$ and $H(\mathfrak{R}^\Omega)$ are elementary equivalent. The statement follows by the previous theorem. \square

Corollary 4.5. *Let S be a set of symbols expanding \mathcal{OS} , and let $\mathfrak{R} = (\mathbb{R}, a)$ be a regular polynomially bounded S -structure with field of exponents Λ . Let $V \subset (\mathbb{R}_{>0})^n$ be a set that is definable in \mathfrak{R} . We denote by $V^{H(\mathfrak{R})}$ the extension of V to $H(\mathfrak{R})$ and by $V^{H(\mathfrak{R}^{\mathbb{R}})}$ the extension of V to $H(\mathfrak{R}^{\mathbb{R}})$. Then*

$$\mathcal{A}_0(V) = \text{Log}(V^{H(\mathfrak{R}^{\mathbb{R}})}).$$

Moreover the subset $\mathcal{A}_0(V) \cap \Lambda^n$ is dense in $\mathcal{A}_0(V)$, and, as $\mathcal{A}_0(V)$ is closed,

$$\mathcal{A}(V^{H(\mathfrak{R})}) = \mathcal{A}(V^{H(\mathfrak{R}^{\mathbb{R}})}) = \text{Log}(V^{H(\mathfrak{R}^{\mathbb{R}})}).$$

Corollary 4.6. *Let $V \subset (\mathbb{R}_{>0})^n$ be a semi-algebraic set. Then $\mathcal{A}_0(V) \cap \mathbb{Q}^n$ is dense in $\mathcal{A}_0(V)$. Let \mathbb{F} be a non-Archimedean real closed field of rank one extending \mathbb{R} , and let $V^{\mathbb{F}}$ be the extension of V to \mathbb{F} . Then*

$$\mathcal{A}_0(V) = \mathcal{A}(V^{\mathbb{F}}).$$

If \mathbb{F} extends $H(\overline{\mathbb{R}}^{\mathbb{R}})$, then

$$\mathcal{A}(V^{\mathbb{F}}) = \text{Log}(V^{\mathbb{F}}).$$

As a further corollary, we prove the following proposition, that will be needed later.

Proposition 4.7. *Let $V \subset (\mathbb{R}_{>0})^n$ be a set definable in a regular polynomially bounded structure, and let $\pi : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be the projection on the first m coordinates (with $m < n$). Then we have*

$$\mathcal{A}_0(\pi(V)) = \pi(\mathcal{A}_0(V)).$$

Proof. We denote by $\pi^{H(\overline{\mathbb{R}})} : H(\overline{\mathbb{R}})^n \rightarrow H(\overline{\mathbb{R}})^m$ the projection on the first m coordinates. The proposition follows easily from Corollary 4.5 and from the fact that $\pi^{H(\overline{\mathbb{R}})}(V^{H(\overline{\mathbb{R}})}) = (\pi(V))^{H(\overline{\mathbb{R}})}$. \square

4.3 Patchworking families. Let S be a set of symbols expanding \mathcal{OS} , and let $\mathfrak{R} = (\mathbb{R}, a)$ be an S -structure expanding $\overline{\mathbb{R}}$. Let \mathbb{F} be a non-Archimedean real closed field of rank one extending \mathbb{R} , and let (\mathbb{F}, b) be an S -structure expanding the \mathcal{OS} structure on the real closed field and extending the S -structure on \mathbb{R} . If V is a definable set in (\mathbb{F}, a) , we have defined the non-Archimedean amoeba $\mathcal{A}(V)$ as the closure in \mathbb{R}^n of the logarithmic image $\text{Log}(V)$. Up to now, we studied the properties of non-Archimedean amoebas only in the particular case in which V is the extension of a definable set in \mathfrak{R} , and this seems an essential requirement because we used extensively the properties of logarithmic limit sets to study non-Archimedean amoebas. This particular case can be called the *constant coefficient case*, because one can consider the elements of \mathbb{F} as real valued functions (this interpretation is clear in the case of Hardy fields) and the elements of \mathbb{R} as constant real valued functions, and the extension to \mathbb{F} of a definable set in \mathfrak{R} is definable with real coefficients.

The purpose of this subsection is to show that with similar methods one can study the properties of general definable sets. We will restrict ourself to the case where (\mathbb{F}, b) is the S -structure $\overline{H(\mathfrak{R})}$ on the Hardy field of \mathfrak{R} . The idea is to define a one parameter family of definable sets in \mathfrak{R} , and to show that the logarithmic limit of this family is the non-Archimedean amoeba. Families constructed with this idea are sometimes called patchworking families in the literature. After having shown that the non-Archimedean amoebas are polyhedral complexes with controlled dimension, we will show how their local structure around a point λ is described by logarithmic limit sets of a particular definable set in \mathfrak{R} that we will call the initial set of λ .

If $V \subset (H(\mathfrak{R})_{>0})^n$ is a definable set, there exist a first order L_S -formula $\varphi(x_1, \dots, x_n, y_1, \dots, y_m)$, and parameters $a_1, \dots, a_m \in H(\mathfrak{R})$ such that

$$V = \{(x_1, \dots, x_n) \mid \varphi(x_1, \dots, x_n, a_1, \dots, a_m)\}.$$

Choose definable functions f_1, \dots, f_m such that $[f_i] = a_i$. These data define a definable set in \mathfrak{R} :

$$\tilde{V} = \{(x_1, \dots, x_n, t) \in (\mathbb{R}_{>0})^{n+1} \mid \varphi(x_1, \dots, x_n, f_1(t), \dots, f_m(t))\}.$$

Suppose that $\varphi'(x_1, \dots, x_n, y_1, \dots, y_{m'})$ is another formula defining V with parameters $a'_1, \dots, a'_{m'}$, and that $f'_1, \dots, f'_{m'}$ are definable functions such that $[f'_i] = a'_i$. These data define

$$\tilde{V}' = \{(x_1, \dots, x_n, t) \in (\mathbb{R}_{>0})^{n+1} \mid \varphi'(x_1, \dots, x_n, f'_1(t), \dots, f'_{m'}(t))\}.$$

As both formulae define V we have

$$\overline{H(\mathfrak{R})} \models \forall x_1, \dots, x_n : \varphi(x_1, \dots, x_n, a_1, \dots, a_m) \iff \varphi'(x_1, \dots, x_n, a'_1, \dots, a'_{m'}).$$

As we said above, we have: $\exists \varepsilon > 0 : \forall t \in (0, \varepsilon) : \mathfrak{R} \models \forall x_1, \dots, x_n :$

$$\varphi(x_1, \dots, x_n, f_1(t), \dots, f_m(t)) \iff \varphi'(x_1, \dots, x_n, f'_1(t), \dots, f'_{m'}(t)).$$

Hence $\tilde{V} \cap (\mathbb{R}^n \times (0, \varepsilon)) = \tilde{V}' \cap (\mathbb{R}^n \times (0, \varepsilon))$, and the set \tilde{V} is “well defined for small enough values of t ”. Actually we prefer to see the set \tilde{V} as a parametrized family:

$$V_t = \{(x_1, \dots, x_n) \in (\mathbb{R}_{>0})^n \mid (x_1, \dots, x_n, t) \in \tilde{V}\}.$$

We can say that the set V determines the germ near zero of this parametrized family. We will use the notation $V_* = (V_t)_{t>0}$ for the family, and we will call these families *patchworking families* determined by V , as they are a generalization of the patchworking families of [24].

Given a patchworking family V_* , we can define the *tropical limit* of the family as

$$\mathcal{A}_0(V_*) = \lim_{t \rightarrow 0} \mathcal{A}_t(V_t) = \lim_{t \rightarrow 0} \text{Log}_t^{-1}(V_t),$$

where the limit is in the sense of subsection 2.1. This is a closed subset of \mathbb{R}^n . Note that this set only depends on V . If V is the extension to $H(\mathfrak{R})$ of a definable subset $W \subset \mathbb{R}^n$, then the patchworking family V_t is constant: $V_t = W$, and the tropical limit is simply the logarithmic limit set: $\mathcal{A}_0(V_*) = \mathcal{A}_0(W)$.

Consider the logarithmic limit set of \tilde{V} :

$$\mathcal{A}_0(\tilde{V}) = \lim_{t \rightarrow 0} \mathcal{A}_t(\tilde{V}) \subset \mathbb{R}^{n+1}.$$

As in subsection 3.2, we consider the set

$$H = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid (x_1, \dots, x_n, -1) \in \mathcal{A}_0(\tilde{V})\}.$$

Note that $\text{Log}_t^{-1}(\mathbb{R}^n \times \{-1\}) = (\mathbb{R}_{>0})^n \times \{t\}$. Hence $\mathcal{A}_0(V_*) = \lim_{t \rightarrow 0} \text{Log}_t^{-1}(V_t) = H$.

Now consider the extension $\tilde{V}^{H(\mathfrak{R})}$ of the \tilde{V} to the Hardy field $H(\mathfrak{R})$. By the results of the previous section, we know that $\mathcal{A}(\tilde{V}^{H(\mathfrak{R})}) = \mathcal{A}_0(\tilde{V})$. If we denote by $t \in H(\mathfrak{R})$ the germ of the identity function, we have that

$$V = \{(x_1, \dots, x_n) \mid (x_1, \dots, x_n, t) \in \tilde{V}^{H(\mathfrak{R})}\},$$

as, for $i \in \{1, \dots, m\}$, we have $f_i(t) = a_i$. Hence, as $\log |t| = -1$, $\mathcal{A}(V) \subset H = \mathcal{A}_0(V_*)$.

Lemma 4.8. $(0, \dots, 0) \in \mathcal{A}_0(V_*) \iff (0, \dots, 0) \in \text{Log}(V)$.

Proof. The forward implication follows from what we said above. The converse follows from the second part of the proof of Lemma 4.1, applied to the set \tilde{V} . \square

Let $\lambda \in \Lambda^n$. We define a twisted set

$$V^\lambda = \{x \in H(\mathfrak{R})^n \mid \varphi(t^{-\lambda_1} x_1, \dots, t^{-\lambda_n} x_n, a_1, \dots, a_m)\}.$$

Then $\lambda \in \text{Log}(V) \Leftrightarrow (0, \dots, 0) \in \text{Log}(V^\lambda)$. Then we define

$$H^\lambda = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid (x_1, \dots, x_n, -1) \in \mathcal{A}_0(\widetilde{V}^\lambda)\}.$$

Now H^λ is simply H translated by the vector $-\lambda$. Hence we get:

Lemma 4.9. *For all $\lambda \in \Lambda$, we have $\lambda \in \mathcal{A}_0(V_*) \Leftrightarrow \lambda \in \text{Log}(V)$.*

Using these facts we can extend the results of the previous sections about logarithmic limit sets and their relations with non-Archimedean amoebas to tropical limits of patchworking families. For example we can prove the following statements.

Theorem 4.10. *Let S be a structure expanding \mathcal{OS} , and let $\mathfrak{R} = (\mathbb{R}, a)$ be a regular polynomially bounded S -structure with field of exponents Λ . Let $V \subset H(\mathfrak{R})$ be a definable set in the S -structure $\overline{H(\mathfrak{R})}$, and let V_* be a patchworking family determined by V . Then the following facts hold:*

- (1) $\mathcal{A}_0(V_*)$ is a polyhedral complex with dimension less than or equal to the dimension of V .
- (2) $\mathcal{A}_0(V_*) \cap \Lambda^n = \text{Log}(V)$.
- (3) $\mathcal{A}_0(V_*) = \mathcal{A}(V)$.
- (4) $\mathcal{A}_0(V_*) \cap \Lambda^n$ is dense in $\mathcal{A}_0(V_*)$.

Proof. Every statement follows from the corresponding statement about logarithmic limit sets, and from the facts exposed above. \square

For every point $\lambda \in \Lambda$, the twisted set V^λ defines a germ of patchworking family $V_*^\lambda = (V_t^\lambda)_{t>0}$, where for every t , V_t^λ is a definable subset of $(\mathbb{R}_{>0})^n$. Consider the set

$$V_0^\lambda = \lim_{t \rightarrow 0} V_t^\lambda,$$

where the limit is defined as in subsection 2.1. The set V_0^λ is a definable subset of $(\mathbb{R}_{>0})^n$, and its logarithmic limit set $\mathcal{A}_0(V_0^\lambda)$ can be used to describe a neighborhood of λ in $\mathcal{A}(V)$, as we will state in the next theorem. The proof follows from Theorem 3.10 applied to the definable set \widetilde{V}^λ . The set V_0^λ , in the language of Theorem 3.10, is called W .

Theorem 4.11. *Let S be a set of symbols expanding \mathcal{OS} , and let $\mathfrak{R} = (\mathbb{R}, a)$ be an S -structure expanding \overline{R} that is o-minimal and polynomially bounded, with field of exponents Λ . Let $V \subset H(\mathfrak{R})$ be a definable set in the S -structure $\overline{H(\mathfrak{R})}$. Then we have*

$$\forall \lambda \in \Lambda^n : \lambda \in \mathcal{A}(V) \iff V_0^\lambda \neq \emptyset.$$

Moreover, if $\Lambda = \mathbb{R}$, for all $\lambda \in \mathbb{R}$, there exists a neighborhood U of λ in $\mathcal{A}(V)$ such that the translation of U by $-\lambda$ is a neighborhood of $(0, \dots, 0)$ in $\mathcal{A}_0(V_0^\lambda)$.

Proof. It follows from the arguments above and from Theorem 3.10. \square

The set V_0^λ is well defined for all λ s, and it depends only on λ . It can be called the *initial set* of λ , as it plays the role of the initial ideal of [17]. The difference is that V_0^λ is a geometric object, while the initial ideal of [17] is an algebraic object.

5 Comparison with other constructions

5.1 Complex algebraic sets. Logarithmic limit sets of complex algebraic sets are a particular case of logarithmic limit sets of real semi-algebraic sets, in the following sense. Let $V \subset \mathbb{C}^n$ be a complex algebraic set, and consider the real semi-algebraic set

$$|V| = \{x \in (\mathbb{R}_{>0})^n \mid \exists z \in V : |z| = x\}.$$

The logarithmic limit set of V as defined in [3] is precisely the logarithmic limit set of $|V|$ in our notation. We will write $\mathcal{A}_0(V) = \mathcal{A}_0(|V|)$. Hence all the results we got about logarithmic limit sets of real semi-algebraic sets produce an alternative proof of the same results for complex algebraic sets, that were originally proved partly in [3] and [4].

Even the description of logarithmic limit sets via non-Archimedean amoebas can be translated to complex algebraic sets. Let \mathbb{F} be a non-Archimedean real closed field of rank one extending \mathbb{R} , and let v be a choice of a real valued valuation on \mathbb{F} , as in subsection 4.1. The field $\mathbb{K} = \mathbb{F}[i]$ is an algebraically closed field extending \mathbb{C} , with an extended valuation $v : \mathbb{K}^* \rightarrow \mathbb{R}$ defined by $v(a + bi) = \min(v(a), v(b))$. The *component-wise logarithm map* can be extended to the field \mathbb{K} by

$$\text{Log} : \mathbb{K}^{*n} \ni (z_1, \dots, z_n) \rightarrow (-v(z_1), \dots, -v(z_n)) \in \mathbb{R}^n.$$

On \mathbb{K} there is also the *complex norm* $|\cdot| : \mathbb{K} \rightarrow \mathbb{F}_{\geq 0}$ defined by $|a + bi| = \sqrt{a^2 + b^2}$. Now if V is an algebraic set in \mathbb{K}^n , the set

$$|V| = \{x \in (\mathbb{F}_{>0})^n \mid \exists z \in V : |z| = x\}$$

is a semi-algebraic set in \mathbb{F}^n . The *logarithmic image* of V is defined as the image $\text{Log}(V)$, and the non-Archimedean amoeba $\mathcal{A}(V)$ is defined as the closure of this image. Clearly, we have $\text{Log}(V) = \text{Log}(|V|)$ and $\mathcal{A}(V) = \mathcal{A}(|V|)$. Moreover, if $V \subset \mathbb{C}^n$ is an algebraic set, and $V^{\mathbb{K}} \subset \mathbb{K}^n$ is its extension to \mathbb{K} , then $(|V|)^{\mathbb{K}} = |V^{\mathbb{K}}|$. These facts directly give the relation between logarithmic limit sets of complex algebraic sets and non-Archimedean amoebas in algebraically closed fields.

The same relation holds with patchworking families. Let $\mathfrak{R} = (\mathbb{R}, a)$ be a regular polynomially bounded structure with field of exponents Λ , let $\mathbb{F} = H(\mathfrak{R})$ and $\mathbb{K} = H(\mathfrak{R})[i]$ and let $V \subset \mathbb{K}^n$ be an algebraic set. There are polynomials $f_1, \dots, f_m \in \mathbb{K}[x_1, \dots, x_n]$ such that $V = V(f_1, \dots, f_m)$. Every polynomial f_j has the form

$$f_j = \sum_{\omega \in \mathbb{Z}^n} (a_{j,\omega} + ib_{j,\omega})x^\omega,$$

where $a_{j,\omega}, b_{j,\omega} \in H(\mathfrak{R})$. Choose representatives functions $\alpha_{j,\omega}, \beta_{j,\omega}$ such that $[\alpha_{j,\omega}] = a_{j,\omega}, [\beta_{j,\omega}] = b_{j,\omega}$. This choice defines families of polynomials

$$f_{j,t} = \sum_{\omega \in S(f)} (\alpha_{j,\omega}(t) + i\beta_{j,\omega}(t))x^\omega$$

and a corresponding family of algebraic sets in \mathbb{C}^n

$$V_t = V(f_{1,t}, \dots, f_{m,t}).$$

We will call these families *patchworking families* because they generalize the patchworking polynomial of [11, Part 2], and we will denote the family by $V_* = (V_t)$. The family V_* depends of the choice of the polynomials f_j and of the definable functions $\alpha_{j,\omega}, \beta_{j,\omega}$. If we change these choices we get another patchworking family coinciding with V_* for $t \in (0, \varepsilon)$. The *tropical limit* of one such family is

$$\mathcal{A}_0(V_*) = \lim_{t \rightarrow 0} \mathcal{A}_t(V_t)$$

where the limit is defined as in subsection 2.1. As before, $|V|$ is a semi-algebraic set in $H(\mathfrak{R})^n$, and if $|V|_* = (|V|_t)$ is a patchworking family defined by $|V|$, then there exists an $\varepsilon > 0$ such that for $t \in (0, \varepsilon)$ we have $|V_t| = |V|_t$. Hence $\mathcal{A}_0(V_*) = \mathcal{A}_0(|V|_*)$, and we can get the properties of the tropical limit of complex patchworking families as a corollary of the properties of tropical limits of real patchworking families.

Let $f \in \mathbb{R}[x_1, \dots, x_n]$. Let V be the intersection of the zero locus of f and $(\mathbb{R}_{>0})^n$, and let $V_{\mathbb{C}}$ be the zero locus of f in \mathbb{C}^n . As $V \subset V_{\mathbb{C}}$, the logarithmic limit set of V is included in the logarithmic limit set of $V_{\mathbb{C}}$. Moreover, as $V_{\mathbb{C}}$ is a complex hypersurface, it is possible to give an easy combinatorial description of $\mathcal{A}_0(V_{\mathbb{C}})$, it is simply the dual fan of the Newton polytope of f . Unfortunately, it is not possible, in general, to use this fact to understand the combinatorics of $\mathcal{A}_0(V)$. There are examples where V is an irreducible hypersurface, and $\mathcal{A}_0(V)$ is a subpolyhedron of $\mathcal{A}_0(V_{\mathbb{C}})$ that is not a subcomplex. For example, consider the real algebraic set W in Figure 6; the logarithmic limit set of W is only the ray in the direction $(-1, 0, 0)$, but this ray lies in the interior of a face of $\mathcal{A}_0(W_{\mathbb{C}})$, for the details see the end of the next subsection.

5.2 Positive tropical varieties. In this subsection we compare the notion of non-Archimedean amoebas for real closed fields that we studied in this paper with a similar object called positive tropical variety studied in [18].

To be consistent with [18], we will denote by $\mathbb{K} = \bigcup_{n=1}^{\infty} \mathbb{C}((t^{1/n}))$ the algebraically closed field of formal Puiseux series with complex coefficients, whose set of exponents is an arithmetic progression of rational numbers, and by $\mathbb{F} = \bigcup_{n=1}^{\infty} \mathbb{R}((t^{1/n}))$ the subfield of series with real coefficients. \mathbb{K} is the algebraic closure of \mathbb{F} . These fields have a natural valuation $v : \mathbb{K} \rightarrow \mathbb{Q}$, with valuation ring \mathcal{O} , and residue map $r : \mathcal{O} \rightarrow \mathbb{C}$. Note that the valuation v is compatible with the order of \mathbb{F} , i.e. the valuation ring $\mathcal{O} \cap \mathbb{F}$ is convex for the order, and that $r(\mathcal{O} \cap \mathbb{F}) = \mathbb{R}$.

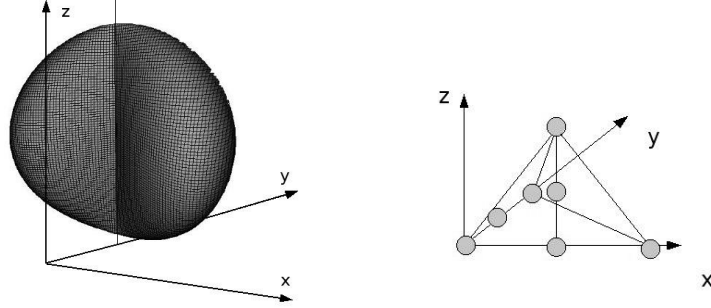


Figure 6. $W = \{(x, y, z) \in \mathbb{R}^3 \mid x^2(1 - (z - 2)^2) = x^4 + (y - 1)^2\}$ is an irreducible surface, but it has a “stick”, the line $\{y = 1, x = 0\}$. The logarithmic limit set of $W \cap (\mathbb{R}_{>0})^3$ is only the ray in the direction $(-1, 0, 0)$, but this ray is contained in the interior part of a face of the dual fan of the Newton polytope of the defining polynomial $x^2(1 - (z - 2)^2) - x^4 - (y - 1)^2$.

We will denote by $\mathbb{F}_{>0}$ the set of positive elements of the field \mathbb{F} . Following [18] we will also use the notation:

$$\mathbb{F}_+ = \{z \in \mathbb{K} \mid r(zt^{-v(z)}) \in \mathbb{R}_{>0}\}.$$

Let V be an algebraic set in \mathbb{K}^n . The set

$$V_{>0} = V \cap (\mathbb{F}_{>0})^n$$

is a semi-algebraic set, whose non-Archimedean amoeba $\mathcal{A}(V_{>0})$, i.e. the closure of the logarithmic image $\text{Log}(V_{>0})$, has been studied in subsection 4.3. In [18] a similar definition is given. The positive part of V is $V_+ = V \cap (\mathbb{F}_+)^n$. The closure of $\text{Log}(V_+)$ is called *positive tropical variety*, and it is denoted by $\text{Trop}^+(V)$. From the definition it is clear that $\mathcal{A}(V_{>0}) \subset \text{Trop}^+(V)$.

In many examples the sets $\mathcal{A}(V_{>0})$ and $\text{Trop}^+(V)$ coincide, but it is also possible to construct examples where the inclusion is strict. For example

$$V = \{(x_1, x_2) \in \mathbb{K}^2 \mid x_1^2 + (x_2 - 1)^2 - x_1^3 = 0\}.$$

Then $V_{>0}$ is the extension to \mathbb{F} of the set in Figure 7, and

$$\begin{aligned} \mathcal{A}(V_{>0}) &= \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 = 0, x_2 \leq 0 \text{ or } x_1 \geq 0, 2x_2 = 3x_1\} \\ \text{Trop}^+(V) &= \mathcal{A}(V_{>0}) \cup \{x_2 = 0, x_1 \leq 0\}. \end{aligned}$$

A more interesting example where $\mathcal{A}(V_{>0}) \subsetneq \text{Trop}^+(V)$ is the following: consider the set

$$W = \{(x, y, z) \in \mathbb{R}^3 \mid x^2(1 - (z - 2)^2) = x^4 + (y - 1)^2\},$$

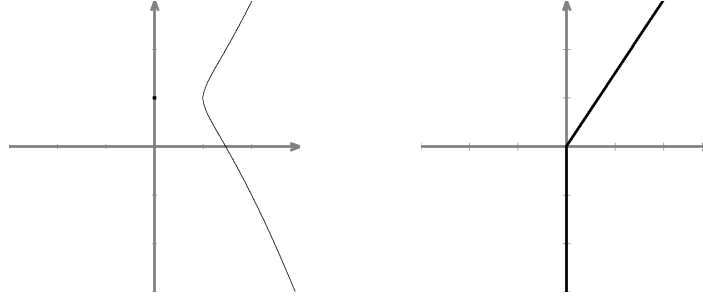


Figure 7. If $V = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 + 1 = 2y + x^3\}$ with an isolated point in $(0, 1)$ (left picture), then $\mathcal{A}_0(V \cap (\mathbb{R}_{>0})^2) = \{(x, y) \in \mathbb{R}^2 \mid x = 0, y \leq 0 \text{ or } x \geq 0, 2y = 3x\}$ (right picture).

the set in Figure 6. This set is a sort of Cartan’s umbrella, an irreducible algebraic set of dimension 2, with a “stick” of dimension 1, the line $\{y = 1, x = 0\}$. Note that the stick is not contained in the positive part $W \cap (\mathbb{R}_{>0})^3$. Given a $t \in (0, 1)$, the amoeba $\mathcal{A}_t(W \cap (\mathbb{R}_{>0})^3)$ goes to infinity only in a neighborhood of the ray in the direction $(-1, 0, 0)$, hence the logarithmic limit set $\mathcal{A}_0(W \cap (\mathbb{R}_{>0})^3)$ is only the ray in the direction $(-1, 0, 0)$. Now define V as the extension of W to \mathbb{F} , $V = W^{\mathbb{F}}$. We have that $\mathcal{A}(V_{>0}) = \mathcal{A}_0(W \cap (\mathbb{R}_{>0})^3)$, the ray in the direction $(-1, 0, 0)$. This ray is in the interior part of a face of $\text{Trop}^+(V)$. Hence not only the two sets do not coincide, but $\mathcal{A}(V_{>0})$ is not a polyhedral subcomplex of $\text{Trop}^+(V)$.

6 Tropical description

Let $V \subset \mathbb{C}^n$ be a complex algebraic variety. The logarithmic limit set $\mathcal{A}_0(V)$ can be described as the intersection of a finite set of tropical zero loci of tropical polynomials. In this section we want to obtain a similar result for logarithmic limit sets of real semi-algebraic sets.

A tropical polynomial is a polynomial over the tropical semifield (for the definition of this \mathcal{OS} -structure, see subsection 6.1 below). Given a polynomial f with complex coefficients, there is a well defined notion of tropicalization, that associates to f a tropical polynomial called the tropicalization of f (see [17]).

There is a notion of tropical zero locus of a tropical polynomial (see [17]), with the property that if f is a polynomial with complex coefficients, the tropical zero locus of the tropicalization of f is equal to the logarithmic limit set of the hypersurface defined by f .

A similar result for logarithmic limit sets of general complex algebraic varieties is more delicate. We say that a finite set of polynomials f_1, \dots, f_s defines V if

$$V = \{z \in \mathbb{C}^n \mid f_1(z) = \dots = f_s(z) = 0\}.$$

If f_1, \dots, f_s form a set of polynomials that defines V , then the logarithmic limit set of V is contained in the intersection of the tropical zero loci of the tropicalizations of the polynomials f_1, \dots, f_s , but sometimes this inclusion is strict. Anyway, a theorem proved in [6] shows that there always exists a finite set of polynomials that defines V with the property that the intersection of the tropical zero loci of the tropicalizations of the polynomials in the set is equal to the logarithmic limit set of V . Such a finite set of polynomials is called a *tropical basis* of V . Tropical bases are a fundamental tool in computational tropical geometry.

In this section we find similar results for logarithmic limit sets of real semi-algebraic sets. Semi-algebraic sets are not simply defined by polynomial equations, one needs to use first order formulae in the language \mathcal{OS} , and we will also use formulae in the language $\mathcal{OS}^{\mathbb{R}}$. Instead of defining the notion of tropicalization of polynomials, we will define the more general notion of dequantization of formulae, see subsection 6.2.

The notion of zero locus of a tropical polynomial is not needed here, that notion is only suitable for complex varieties, or for varieties defined over algebraically closed fields. Here, in the real setting, we will use the different but very natural notion of set defined by a dequantized formula to replace it.

In Section 6.3 we show that if the formula is a positive formula (see below) defining V , then $\mathcal{A}_0(V)$ is contained in the set defined by the dequantized formula. Positive formulae are very good ways to describe closed semi-algebraic sets.

Finally, in subsection 6.4, we will show that if V is a closed semi-algebraic set, then there exists a positive formula such that the set defined by the dequantization of this formula is equal to $\mathcal{A}_0(V)$. In a sense, this result is the analog, for semi-algebraic sets, of the existence of a tropical basis.

6.1 Maslov dequantization. Every real number $t \in (0, 1)$ defines an analytic function:

$$\mathbb{R}_{>0} \ni z \longrightarrow \log_{(\frac{1}{t})} z = \left(\frac{-1}{\log_e t} \right) \log_e z \in \mathbb{R}.$$

This function is bijective, with inverse $x \longrightarrow t^{-x}$, and it preserves the order \leq . The operations ('+' and '\cdot') are transformed via conjugation in the following way:

$$\begin{aligned} x \oplus_t y &= \log_{(\frac{1}{t})}(t^{-x} + t^{-y}) \\ x \odot_t y &= \log_{(\frac{1}{t})}(t^{-x} \cdot t^{-y}) = x + y \end{aligned}$$

Hence every t induces an \mathcal{OS} -structure on \mathbb{R} :

$$\mathbb{R}^t = (\mathbb{R}, \{\leq\}, \{\oplus_t, \odot_t\}, \emptyset).$$

This structure is isomorphic to $\mathbb{R}_{>0}$, hence it is an ordered semifield. In the limit for t tending to zero we have $\lim_{t \rightarrow 0^+} x \oplus_t y = \max(x, y)$.

The limit \mathcal{OS} -structure is called the *tropical semifield*:

$$\mathbb{R}^{\text{trop}} = (\mathbb{R}, \{\leq\}, \{\max, +\}, \emptyset).$$

This is again an ordered semifield, we will denote its operations by $\oplus = \max$ and $\odot = +$. Note the inequality

$$x_1 \oplus \cdots \oplus x_n \leq x_1 \oplus_t \cdots \oplus_t x_n \leq (x_1 \oplus \cdots \oplus x_n) + \log_{\frac{1}{t}} n.$$

In other words the convergence of the family \mathbb{R}^t to the structure \mathbb{R}^{trop} is uniform. This construction is usually called *Maslov dequantization*.

Note that if $\alpha \in \mathbb{R}_{>0}$, the function $\mathbb{R}_{>0} \ni x \rightarrow x^\alpha \in \mathbb{R}_{>0}$ is transformed, via conjugation with the map $\log_{\frac{1}{t}}$, in the map

$$\mathbb{R} \ni x \rightarrow \log_{\frac{1}{t}}((t^{-x})^\alpha) = \alpha x.$$

As this map does not depend on t , it induces also a map in the limit structure \mathbb{R}^{trop} . With these maps, \mathbb{R}^t and \mathbb{R}^{trop} become $\mathcal{OS}^{\mathbb{R}}$ -structures.

The family of maps Log_t , which we used to construct the logarithmic limit sets, is the Maslov dequantization applied coordinate-wise to $(\mathbb{R}_{>0})^n$.

6.2 Dequantization of formulae. An $L_{\mathcal{OS}^{\mathbb{R}}}$ -term $u(x_1, \dots, x_n, y_1, \dots, y_m)$ (see [9, Chapter II, Definition 3.1]) and the constants $a_1, \dots, a_m \in \mathbb{R}_{>0}$ define a function

$$U : (\mathbb{R}_{>0})^n \ni (x_1, \dots, x_n) \rightarrow u(x_1, \dots, x_n, a_1, \dots, a_m) \in \mathbb{R}_{>0}.$$

For every t , this function defines, by conjugation with the map $\log_{\frac{1}{t}}$, a function on \mathbb{R}^n corresponding to the term u where the operations are interpreted with the operations of \mathbb{R}^t , and every constant a_i is interpreted as $\log_{\frac{1}{t}}(a_i)$:

$$U_t = \log_{\frac{1}{t}} \circ U \circ (\text{Log}_{\frac{1}{t}})^{-1} : \mathbb{R}^n \rightarrow \mathbb{R}.$$

Lemma 6.1. *Let $U_0 : \mathbb{R}^n \rightarrow \mathbb{R}$ be the function defined by the term u where the operations are interpreted with the operations of \mathbb{R}^{trop} , and every constant a_i is interpreted as 0. Then*

$$\forall x \in \mathbb{R}^n : U_0(x) \leq U_t(x) \leq U_0(x) + \log_{\frac{1}{t}} C,$$

where C is a constant depending only on the term u and the coefficients a_i . In particular the family of functions U_t uniformly converges to the function U_0 .

Proof. By induction on the complexity of the term. If $u = x_1$, then $U_0 = U_t$ and $C = 1$. If $u = y_1$ then $U_t = \log_{\frac{1}{t}} a_1$ and $U_0 = 0$, hence $C = a_1$. If $u = v^\alpha$, where $\alpha \in \mathbb{R}$, then $V_0 \leq V_t \leq V_0 + \log_{\frac{1}{t}} C$, hence $U_0 \leq U_t \leq U_0 + \log_{\frac{1}{t}} C^\alpha$. If $u = v \cdot w$, then $V_0 \leq V_t \leq V_0 + \log_{\frac{1}{t}} C$ and $W_0 \leq W_t \leq W_0 + \log_{\frac{1}{t}} D$, hence $U_0 \leq U_t \leq U_0 + \log_{\frac{1}{t}} CD$. If $u = v + w$, then $V_0 \leq V_t \leq V_0 + \log_{\frac{1}{t}} C$ and $W_0 \leq W_t \leq W_0 + \log_{\frac{1}{t}} D$, hence $U_0 \leq U_t \leq U_0 + \log_{\frac{1}{t}} 2 \max(C, D)$. \square

If $\varphi(x_1, \dots, x_n, y_1, \dots, y_m)$ is an $L_{\mathcal{OS}^{\mathbb{R}}}$ -formula and a_1, \dots, a_m are constants, they define the set

$$V = \{(x_1, \dots, x_n) \in (\mathbb{R}_{>0})^n \mid \varphi(x_1, \dots, x_n, a_1, \dots, a_m)\}.$$

We will denote by φ_t the formula φ where the operations are interpreted in the structure \mathbb{R}^t , and φ_0 the formula φ where the operations are interpreted in the structure \mathbb{R}^{trop} . Hence

$$\mathcal{A}_t(V) = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid \varphi_t(x_1, \dots, x_n, \log_{\frac{1}{t}} a_1, \dots, \log_{\frac{1}{t}} a_m)\}.$$

Because $\log_{\frac{1}{t}}$ is a semifield isomorphism, the amoeba $\mathcal{A}_t(V)$ is described by the same formula. Anyway it is not always true that

$$\mathcal{A}_0(V) = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid \varphi_0(x_1, \dots, x_n, 0, \dots, 0)\}.$$

For example if $\varphi(x_1) = \neg(x \leq 1)$, then $\varphi_0 = \neg(x \leq 0)$, but the logarithmic limit set of $\{x > 1\}$ is not $\{x > 0\}$, but $\{x \geq 0\}$.

6.3 Dequantization of sets. A *positive formula* is a formula written without the symbols $\neg, \Rightarrow, \Leftrightarrow$. These formulae contain only the connectives \vee and \wedge and the quantifiers \forall, \exists . Consider the standard $\mathcal{OS}^{\mathbb{R}}$ -structure on $\mathbb{R}_{>0}$, or one of the $\mathcal{OS}^{\mathbb{R}}$ -structures \mathbb{R}^t or \mathbb{R}^{trop} on \mathbb{R} . Every subset of $(\mathbb{R}_{>0})^n$ or \mathbb{R}^n that is defined by a quantifier-free positive $L_{\mathcal{OS}^{\mathbb{R}}}$ -formula in one of these structures is closed, as the set of symbols $\mathcal{OS}^{\mathbb{R}}$ has only the relations $=$ and \leq , that are closed, and the functions $+, \cdot, x^\alpha$, that are continuous.

Proposition 6.2. *Let $\varphi(x_1, \dots, x_n, y_1, \dots, y_m)$ be a positive $L_{\mathcal{OS}^{\mathbb{R}}}$ -formula, and let $a_1, \dots, a_m \in \mathbb{R}_{>0}$ be parameters. If V is such that*

$$V \subset \{(x_1, \dots, x_n) \in (\mathbb{R}_{>0})^n \mid \varphi(x_1, \dots, x_n, a_1, \dots, a_m)\},$$

then

$$\mathcal{A}_0(V) \subset \{x \in \mathbb{R}^n \mid \varphi_0(x_1, \dots, x_n, 0, \dots, 0)\}.$$

Proof. By induction on the complexity of the formula. If φ is atomic, then it has the form $u(x_1, \dots, x_n, y_1, \dots, y_m) \mathcal{R} v(x_1, \dots, x_n, y_1, \dots, y_m)$, where \mathcal{R} is $=$ or \leq . We have

$$\mathcal{A}_t(V) \subset \{x \in \mathbb{R}^n \mid \varphi_t(x_1, \dots, x_n, \log_{\frac{1}{t}}(a_1), \dots, \log_{\frac{1}{t}}(a_m))\}.$$

We may put all the equations together, one for every t , thus finding a description for the deformation

$$\mathcal{D} = \{(x, t) \in \mathbb{R}^n \times (0, \varepsilon) \mid x \in \mathcal{A}_t(V)\}$$

$$\mathcal{D} = \{(x_1, \dots, x_n, t) \in \mathbb{R}^n \times (0, 1) \mid \varphi_t(x_1, \dots, x_n, \log_{\frac{1}{t}}(a_1), \dots, \log_{\frac{1}{t}}(a_m))\}.$$

If we consider U_t and V_t as functions on $\mathbb{R}^n \times (0, 1)$, they can be extended continuously to $\mathbb{R}^n \times [0, 1)$ defining the extensions on $\mathbb{R}^n \times \{0\}$ by U_0, V_0 . Hence we get the following inclusion for the logarithmic limit set:

$$\mathcal{A}_0(V) \subset \{x \in \mathbb{R}^n \mid \varphi_0(x_1, \dots, x_n, 0, \dots, 0)\}.$$

If $\varphi = \psi_1 \vee \psi_2$ (respectively $\varphi = \psi_1 \wedge \psi_2$), then $V \subset V_1 \cup V_2$ (respectively $V \subset V_1 \cap V_2$), where V_i is defined by ψ_i . The statement follows from Proposition 2.2.

If $\varphi = \exists x_{n+1} : \psi$, then V is contained in the projection of W , where W is the set defined by ψ . The statement follows from Proposition 4.7.

If $\varphi = \forall x_{n+1} : \psi$, then we denote by W the set defined by ψ . If $(0, \dots, 0, -1) \in \mathcal{A}_0(V)$, there is a sequence $x(k)$ in V converging to a point $(b_1, \dots, b_{n-1}, 0)$ with $b_i \neq 0$. Then W contains a sequence of lines $\{(x(k), y) \mid y \in \mathbb{R}_{>0}\}$, hence $\mathcal{A}_0(W)$ contains the line $\{(0, \dots, 0, y) \mid y \in \mathbb{R}\}$. As $\mathcal{A}_0(W) \subset \{(x_1, \dots, x_{n+1}) \mid \psi_0(x_1, \dots, x_{n+1})\}$, then $\mathcal{A}_0(V) \subset \{(x_1, \dots, x_n) \mid \forall x_{n+1} : \psi_0(x_1, \dots, x_{n+1})\}$. \square

There are examples where $V = \{x \in (\mathbb{R}_{>0})^n \mid \varphi(x_1, \dots, x_n, a_1, \dots, a_m)\}$ with φ a positive $L_{\mathcal{OS}^{\mathbb{R}}}$ -formula, and

$$\mathcal{A}_0(V) \subsetneq \{x \in \mathbb{R}^n \mid \varphi_0(x_1, \dots, x_n, 0, \dots, 0)\}.$$

For example consider the following atomic formula

$$\varphi(x_1, x_2, y_1, y_2, y_3) : x_1^2 + x_2^2 + y_1 = y_2 x_1 + y_3 x_2,$$

with constants $a_1 = 13 - r^2, a_2 = 4, a_3 = 6$, with $r^2 < 13$. This is the equation of a circle, $(x_1 - 2)^2 + (x_2 - 3)^2 = r^2$. The dequantized formula does not depend on the value of r :

$$\varphi_0(x_1, x_2, 0, 0, 0) : \max(2x_1, 2x_2, 0) = \max(x_1, x_2).$$

Now if $V_r = \{(x_1, x_2) \in (\mathbb{R}_{>0})^2 \mid x_1^2 + x_2^2 + (13 - r^2) = 4x_1 + 6x_2\}$ the logarithmic limit sets of $V_{\frac{3}{2}}, V_{\frac{5}{2}}, V_{\frac{7}{2}}$ are different (see Figure 8): $\mathcal{A}_0(V_{\frac{3}{2}})$ is the single point $(0, 0)$, because $V_{\frac{3}{2}}$ is compact, $\mathcal{A}_0(V_{\frac{5}{2}})$ is the ray in the direction $(-1, 0)$, and $\mathcal{A}_0(V_{\frac{7}{2}})$ is the union of the two rays in the directions $(-1, 0)$ and $(0, -1)$. For $V_{\frac{7}{2}}$, the formula φ_0 gives a definition of the logarithmic limit set:

$$\mathcal{A}_0(V_{\frac{7}{2}}) = \{(x_1, x_2) \in \mathbb{R}^2 \mid \max(2x_1, 2x_2, 0) = \max(x_1, x_2)\},$$

but for $\mathcal{A}_0(V_{\frac{3}{2}})$ and $\mathcal{A}_0(V_{\frac{5}{2}})$ we have a strict inclusion.

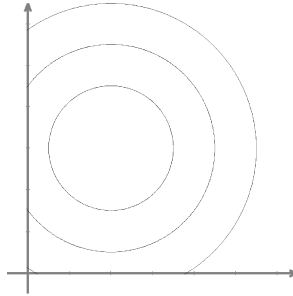


Figure 8. $V_r = \{(x, y) \in (\mathbb{R}_{>0})^2 \mid x^2 + y^2 + 13 = r^2 + 4x + 6y\}$ for $r = \frac{3}{2}, \frac{5}{2}, \frac{7}{2}$ respectively. Their logarithmic limit sets are different.

Even if $\varphi(x_1, x_2, \frac{27}{4}, 4, 6)$ is a definition of $V_{\frac{5}{2}}$, $\varphi_0(x_1, x_2, 0, 0, 0)$ is not a definition of $\mathcal{A}_0(V_{\frac{5}{2}})$. Anyway we can find another formula with this property, for example:

$$(4x_1^3 + 16x_1^2 + 37x_1 + 40)\left(x_1^2 - 4x_1 + x_2^2 - 6x_2 + \frac{27}{4}\right) = 0.$$

This formula gives a definition of $V_{\frac{5}{2}}$ because it is the product of a polynomial defining $V_{\frac{5}{2}}$ by a polynomial that has no positive zeros. The dequantized version of this formula is $\max(5x_1, 0, 2x_2, 3x_1 + 2x_2) = x_2 + \max(3x_1, 0)$, and one can check explicitly that this formula defines the ray in the direction $(-1, 0)$, i.e. it is an exact description of $\mathcal{A}_0(V_{\frac{5}{2}})$.

There are examples of real algebraic sets V where it is not possible to find an algebraic formula φ defining V such that φ_0 is a definition of $\mathcal{A}_0(V)$. Here by an algebraic formula we mean an atomic formula with the relation $=$, in other words an equation between two positive polynomials.

Consider for example the cubic $V = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1^2 + x_2^2 + 1 = 2x_2 + x_1^3\}$, as in Figure 7. This cubic has an isolated point in $(0, 1)$. This point is outside the positive orthant $(\mathbb{R}_{>0})^2$, hence it does not influence the logarithmic limit set of $V' = V \cap (\mathbb{R}_{>0})^2$, but the set defined by $\{(x_1, x_2) \in \mathbb{R}^2 \mid \max(2x_1, 2x_2, 0) = \max(x_2, 3x_1)\}$ contains also the half line $\{x_2 = 0, x_1 \leq 0\}$ that is not in the logarithmic limit set, and the same happens for every polynomial equation defining V .

We need to use the order relation \leq to construct a formula φ defining V' such that φ_0 is a definition of $\mathcal{A}_0(V')$. For example:

$$V' = \left\{ (x_1, x_2) \in (\mathbb{R}_{>0})^2 \mid x_1^2 + x_2^2 + 1 = 2x_2 + x_1^3 \wedge x_1 \geq \frac{1}{2} \right\}$$

$$\mathcal{A}_0(V') = \left\{ (x_1, x_2) \in \mathbb{R}^2 \mid \max(2x_1, 2x_2, 0) = \max(x_2, 3x_1) \wedge x_1 \geq 0 \right\}.$$

As we will see in the next subsection, this is a general fact.

6.4 Exact definition. Let $C \subset \mathbb{R}^n$ be an open convex set such that $(0, \dots, 0, -1) \in C$ and the closure \overline{C} is a convex polyhedral cone contained in $\{x \in \mathbb{R}^n \mid x_n < 0\} \cup \{0\}$. The faces F_1, \dots, F_k of \overline{C} are described by equations

$$a_i^1 x_1 + \dots + a_i^{n-1} x_{n-1} + x_n = 0$$

and C is described by

$$C = \{x \in \mathbb{R}^n \mid x_n < 0 \text{ and } \forall i \in \{1, \dots, k\} : a_i^1 x_1 + \dots + a_i^{n-1} x_{n-1} + x_n < 0\}.$$

For every $h \in \mathbb{R}_{>0}$, consider the set

$$E_h(C) = \{x \in (\mathbb{R}_{>0})^n \mid x_n < h \text{ and } \forall i \in \{1, \dots, k\} : x_1^{a_i^1} \dots x_{n-1}^{a_i^{n-1}} x_n < h\}.$$

Lemma 6.3. *Let $V \subset (\mathbb{R}_{>0})^n$ be a set such that $\mathcal{A}_0(V) \cap C = \emptyset$. Then for every sufficiently small $h \in \mathbb{R}_{>0}$ we have $V \cap E_h(C) = \emptyset$.*

Proof. Suppose that for all $i \in \mathbb{N}$ there exists $x_i \in V \cap E_{\frac{1}{i}}(C)$. Then from the sequence $(x_i) \subset V$ we can extract a subsequence y_i such that $\text{Log}_e(y_i)$ converges to a point $y \in C$. \square

Note that $E_h(C)$ is described by the following $L_{OS^{\mathbb{R}}}$ -formula, with $y = h$

$$\varphi^C(x_1, \dots, x_n, y) = \neg(y \leq x_n \vee y \leq x_1^{a_1^1} \dots x_{n-1}^{a_{n-1}^{n-1}} x_n \vee \dots \vee y \leq x_1^{a_1^n} \dots x_{n-1}^{a_{n-1}^{n-1}} x_n)$$

and C is described by the formula φ_0^C with $y = 0$.

Let $C \subset \mathbb{R}^n$ be an open convex set such that the closure \overline{C} is a convex polyhedral cone and $\overline{C} \subset H \cup \{0\}$ where H is an open half-space. There exists a linear map B such that $(0, \dots, 0, -1) \in B(C)$, and $B(\overline{C})$ is contained in $\{x \in \mathbb{R}^n \mid x_n < 0\} \cup \{0\}$. We will use the notation

$$E_h(C) = \overline{B}^{-1}(E_h(B(C))).$$

As before there exists an $L_{OS^{\mathbb{R}}}$ -formula $\varphi^C(x_1, \dots, x_n, y)$ such that

$$\begin{aligned} E_h(C) &= \{x \in (\mathbb{R}_{>0})^n \mid \varphi^C(x_1, \dots, x_n, h)\} \\ C &= \{x \in \mathbb{R}^n \mid \varphi_0^C(x_1, \dots, x_n, 0)\}. \end{aligned}$$

Let $V \subset (\mathbb{R}_{>0})^n$ be a set definable in an o-minimal, polynomially bounded structure with field of exponents \mathbb{R} . Then, by Theorem 3.11, $\mathcal{A}_0(V)$ is a polyhedral complex, hence we can find a finite number of sets C_1, \dots, C_k such that

- (1) $C_1 \cup \dots \cup C_k$ is the complement of $\mathcal{A}_0(V)$.
- (2) The closure $\overline{C_i}$ is a convex polyhedral cone.
- (3) There exists an open half-space H_i such that $\overline{C_i} \subset H_i \cup \{0\}$.

Lemma 6.4. *Consider the $L_{OS^{\mathbb{R}}}$ -formula*

$$\varphi(x_1, \dots, x_n, y) = \neg\varphi^{C_1}(x_1, \dots, x_n, y) \wedge \dots \wedge \neg\varphi^{C_k}(x_1, \dots, x_n, y).$$

Then

$$\mathcal{A}_0(V) = \{x \in \mathbb{R}^n \mid \varphi_0(x_1, \dots, x_n, 0)\}$$

and for every sufficiently small $h \in \mathbb{R}_{>0}^n$ we have

$$V \subset \{x \in (\mathbb{R}_{>0})^n \mid \varphi(x_1, \dots, x_n, h)\}.$$

Proof. The first assertion is trivial, and the second assertion follows from the previous lemma. \square

Note that the formula φ of the previous lemma has the form $\psi_1 \wedge \dots \wedge \psi_k$, where the ψ_i have the form

$$\psi_i = y \leq x_1^{a_1^1} \dots x_n^{a_n^1} \vee \dots \vee y \leq x_1^{a_1^m} \dots x_n^{a_n^m}.$$

These formulae do not contain the $+$ operation, hence when they are interpreted with the dequantizing operations \oplus_t, \odot_t or the tropical operations \oplus, \odot the interpretation does not depend on t , and it is simply

$$\psi_i = y \leq a_1^1 x_1 + \dots + a_n^1 x_n \vee \dots \vee y \leq a_1^m x_1 + \dots + a_n^m x_n.$$

Corollary 6.5. *Let V be definable in an o-minimal, polynomially bounded structure with field of exponents \mathbb{R} . For $\varepsilon > 0$, and for small enough $t > 0$, we have*

$$\sup_{x \in \mathcal{A}_t(V)} d(x, \mathcal{A}_0(V)) < \varepsilon.$$

Proof. Choose h such that $V \subset \{\varphi(x_1, \dots, x_n, h)\}$. Then $\mathcal{A}_t(V) \subset \{\varphi_t(x_1, \dots, x_n, \log_{\frac{1}{t}} h)\}$. Note that $\{\varphi_t(x_1, \dots, x_n, \log_{\frac{1}{t}} h)\}$ is a uniformly bounded neighborhood of $\mathcal{A}_0(V)$, with distance depending linearly on y , hence the distance tends to zero when y tends to zero. \square

Theorem 6.6. *Let $\varphi(x_1, \dots, x_n, y_1, \dots, y_m)$ be a positive $L_{\mathcal{OS}^{\mathbb{R}}}$ -formula, let $a_1, \dots, a_m \in \mathbb{R}_{>0}$ be parameters and let*

$$V = \{(x_1, \dots, x_n) \in (\mathbb{R}_{>0})^n \mid \varphi(x_1, \dots, x_n, a_1, \dots, a_m)\}.$$

Then there exists a positive $L_{\mathcal{OS}^{\mathbb{R}}}$ -formula $\psi(x_1, \dots, x_n, y_1, \dots, y_l)$ and parameters $b_1, \dots, b_l \in \mathbb{R}_{>0}$ such that

$$\begin{aligned} V &= \{(x_1, \dots, x_n) \in (\mathbb{R}_{>0})^n \mid \psi(x_1, \dots, x_n, b_1, \dots, b_l)\}, \\ \mathcal{A}_0(V) &= \{x \in \mathbb{R}^n \mid \psi_0(x_1, \dots, x_n, 0, \dots, 0)\}. \end{aligned}$$

Proof. Let $\varphi'(x_1, \dots, x_n, y)$ and h as in Lemma 6.4. Then $\psi = \varphi \wedge \varphi'$ is the searched formula. \square

Corollary 6.7. *Let $V \subset (\mathbb{R}_{>0})^n$ be a closed semi-algebraic set. Then there exists a positive quantifier-free $L_{\mathcal{OS}^{\mathbb{R}}}$ -formula $\varphi(x_1, \dots, x_n, y_1, \dots, y_m)$ and constants $a_1, \dots, a_m \in \mathbb{R}_{>0}$ such that*

$$\begin{aligned} V &= \{x \in (\mathbb{R}_{>0})^n \mid \varphi(x_1, \dots, x_n, a_1, \dots, a_m)\}, \\ \mathcal{A}_0(V) &= \{x \in \mathbb{R}^n \mid \varphi_0(x_1, \dots, x_n, 0, \dots, 0)\}. \end{aligned}$$

Proof. By [5, Theorem 2.7.2], every closed semi-algebraic set is defined by a positive quantifier-free $L_{\mathcal{OS}}$ -formula. \square

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