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A tropical compactification for character spaces of convex projective structures

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Introduction

The original construction of the boundary of the Teichmüller space of an hyperbolic surface S was made by Thurston using mostly techniques from hyperbolic geometry. He saw the points of the Teichmüller space as projectivized length spectra of hyperbolic structures on S, and he compactified the space by adding, as a boundary, the projectivized spectra of measured laminations on S, that played the role of degenerated hyperbolic structures. The action of the mapping class group of the surface on the Teichmüller space extended continuously to an action on the boundary. Thurston showed that this action on the boundary preserved a natural piecewise linear structure on it.

Later, Morgan and Shalen constructed the same boundary using different techniques, mostly from algebra and complex algebraic geometry (see [MS84], [MS88], [MS88']). In their construction they identify the Teichmüller space with a connected component of the real part of a complex algebraic set, namely the variety of all characters of representations of the fundamental group of the surface in $SL_2(\mathbb{C})$. This variety was generated by the trace functions I_γ, $\gamma \in \pi_1(S)$, and the compactification was made by taking the limit points of the ratios of the values $[\log(|I_\gamma(x)| + 2)]_{\gamma \in \pi_1(S)}$. With every boundary points they associated a valuation of the field of fractions of an irreducible component of the character variety, and this valuation defined in a natural way an action of $\pi_1(S)$ on a real tree (a generalization of an ordinary simplicial tree), whose projectivized spectrum corresponded to the boundary point. They also showed that every action of $\pi_1(S)$ on a real tree induced, dually, a measured lamination on S, recovering the interpretation of Thurston.

Note that the Teichmüller space has the structure of connected component of a real algebraic set, while the boundary has a piecewise linear structure. We can see a similar relation between the objects parametrized by these spaces: interior points are parameters for Riemann surfaces that can be seen as complex algebraic curves, while the boundary points parametrize actions of $\pi_1(S)$ on real trees, a generalization of simplicial trees, that are polyhedral objects. Hence both the construction of the boundaries and the interpretation of boundary points seem to be the effect of a degeneration from an algebraic object to a polyhedral object. Moreover the logarithms of polynomial functions degenerate to valuations.

This recalls directly some features of tropical geometry, the geometry of the tropical semifield, a semifield $\mathbb{T}$ with $\mathbb{R}$ as the underlying set and max and + as, respectively, addiction and multiplication. Polynomials (with several variables) in this semifield are convex piecewise linear functions, and their tropical "zero loci" are polyhedral subsets of $\mathbb{R}^n$. Moreover there is a deformation, the Maslov dequantization, of the semifield $\mathbb{R}_{>0}$ in the tropical semifield. This deformation is made by taking logarithms of real numbers with increasing bases. Variations of this construction have been used by Viro ([Vi]) and Mikhalkin ([Mi]) to describe the combinatorial patchworking theorem, and what they do is based on the fact that by Maslov dequantization an algebraic variety (in the usual sense) can be deformed in a tropical variety, a polyhedral set. A tropical variety can be described as the image, through the componentwise valuation map, of an algebraic variety over a non-archimedean field.

In this thesis we started to study in a systematic way the similarities between tropical geometry and the construction of compactification of Teichmüller spaces. We developed a general construction, in the framework of tropical geometry, that, in the particular case of Teichmüller spaces gives the Thurston boundary, with its piecewise linear structure, and the interpretation of boundary point as actions of $\pi_1(S)$ on real trees. Then we searched for other spaces of geometric structures that could be compactified with the same techniques, and we chose the spaces of strictly convex projective structures on a manifold M, spaces that shares many properties with Teichmüller spaces. We constructed a boundary for these spaces and we interpreted the boundary points as actions of $\pi_1(M)$ on tropical projective spaces.

The work starts with a description of the variety of characters of representations of a finitely generated group Γ in $SL_n(\mathbb{K})$. The space of all representations $\mathrm{Hom}(\Gamma, SL_n(\mathbb{K}))$ is an affine algebraic set. If $\mathbb{K} = \mathbb{C}$, it follows from the theory in [MFK94] and [Pr76] that the set of all characters of representations in $\mathrm{Hom}(\Gamma, SL_n(\mathbb{C}))$ is an affine algebraic set $\mathrm{Char}(\Gamma, SL_n(\mathbb{C}))$ with a natural map $t : \mathrm{Hom}(\Gamma, SL_n(\mathbb{C})) \longrightarrow \mathrm{Char}(\Gamma, SL_n(\mathbb{C}))$ associating with each representation its character. We need similar results when $\mathbb{K} = \mathbb{R}$. In this case, if we denote by $\mathrm{Char}(\Gamma, SL_n(\mathbb{R}))$ the real part of $\mathrm{Char}(\Gamma, SL_n(\mathbb{C}))$, the map $t : \mathrm{Hom}(\Gamma, SL_n(\mathbb{R})) \longrightarrow \mathrm{Char}(\Gamma, SL_n(\mathbb{R}))$ is not surjective, hence the affine algebraic set $\mathrm{Char}(\Gamma, SL_n(\mathbb{R}))$ is not in bijection with the set of all characters of representations in $\mathrm{Hom}(\Gamma, SL_n(\mathbb{R}))$. We prove that the image of t is closed, identifying the set of characters with a closed semi-algebraic subset $\overline{\mathrm{Char}}(\Gamma, SL_n(\mathbb{R}))$, and that the image through t of every closed (open) conjugation-invariant semi-algebraic subset of $\mathrm{Hom}(\Gamma, SL_n(\mathbb{R}))$ is closed (open) in $\overline{\mathrm{Char}}(\Gamma, SL_n(\mathbb{R}))$ (theorem 11 and corollary 12). See chapter 1 for details.

Then we review some of the theory of geometric structures on manifolds, focusing on the facts needed to prove that the Teichmüller space of a surface S is a closed semi-algebraic subset of $\overline{\mathrm{Char}}(\pi_1(S), SL_2(\mathbb{R}))$, and the space of strictly convex projective structures on an n-manifold M is, under some hypotheses on $\pi_1(M)$, a closed semi-algebraic subsets of $\overline{\mathrm{Char}}(\pi_1(M), SL_{n+1}(\mathbb{R}))$, $n \geq 2$. This last fact is based on some deep results of Benoist ([Be1], [Be2], [Be3]) and on corollary 12. See chapter 2 for details.

Once we know that the spaces we are interested in are semi-algebraic, we need to understand what happens when we apply the Maslov dequantization to a real semi-algebraic set. The resulting set is called logarithmic limit set. The logarithmic limit sets of complex algebraic sets are now usually called tropical varieties, they have been studied extensively (see [Be71], [BG81], [SS04], [EKL06] and [BJSST07]), using also other names like Bergman fans, Bergman sets, Bieri-Groves sets or non-archimedean amoebas. In this case the logarithmic limit set is a polyhedral complex of the same dimension as the algebraic set, it is described by tropical equations and it is the image, under the component-wise valuation map, of an algebraic set over an algebraically closed non-archimedean field. We extend these results to the logarithmic limit sets of real algebraic and semi-algebraic sets, using techniques coming from o-minimal geometry and model theory. We can prove that the logarithmic limit sets of real semi-algebraic sets (and of sets definable in some o-minimal structures) are polyhedral complexes with dimension less than or equal to the dimension of the semi-algebraic set, and they are the image, under the component-wise valuation map, of an extension of the semi-algebraic set to a carefully chosen real closed non-archimedean field, the Hardy field H of a particular o-minimal structure on $\mathbb{R}$. An analysis of the defining equations and inequalities is carried out, showing that the logarithmic limit set of a closed semi-algebraic set can be described by applying the Maslov dequantization to a suitable formula defining the semi-algebraic set. See chapters 3 and 4 for details.

Then we apply the Maslov dequantization to the Teichmüller space of a surface S and to the space of convex projective structures on a manifold M, satisfying some hypotheses on $\pi_1(M)$. More precisely, we construct an inverse system of logarithmic limit sets, whose inverse limit we use to construct a compactification. This inverse limit is a cone, and it is the image of an extension of the space to a non-archimedean field. For the Teichmüller spaces, the boundary constructed in this way is the Thurston boundary, endowed with a natural piecewise linear structure, that is equivalent to the one defined by Thurston. This shows how the piecewise linear structure on the boundary is induced by the semi-algebraic structure on the interior part. With the same techniques we construct a compactification for the spaces of convex projective structures on a manifold. See chapter 5 for details.

Finally we investigate which objects can be used for the interpretation of the boundary points. Points of the interior part of $\overline{\mathrm{Char}}(\pi_1(M), SL_{n+1}(\mathbb{R}))$ correspond to representations of $\pi_1(M)$ on $SL_{n+1}(\mathbb{R})$, or, geometrically, to actions of the group on a projective space of dimension n. Points of the boundary correspond instead to representations of the group in $SL_{n+1}(\mathbb{K})$, where $\mathbb{K}$ is a non-archimedean field. We find a geometric interpretation of these representations, as actions of the group on tropical projective spaces of dimension n. Note that there exists a naif notion of tropical projective space, the projective quotient of a free module $\mathbb{T}^n$, but these spaces have few invertible projective maps, hence they have few group actions. We give a more general notion of tropical modules and, correspondingly, of tropical projective spaces. We show that these objects have an intrinsic metric, the tropical version of the Hilbert metric. This metric is invariant for tropical projective maps, and the the topology induced by it is contractible. Then we construct a special class of tropical projective spaces by using a generalization of the Bruhat-Tits building for SL_n to non-archimedean fields with a surjective real valuation. See chapter 6 for details.

For the Teichmüller spaces, the boundary points are interpreted as actions of the fundamental group of the surface on real trees as in [MS84]. For the space of convex projective structures we get, instead, a n-dimensional generalization of real trees, with a metric and a structure of tropical projective space. We also show that every action of $\pi_1(M)$ on a tropical projective space has an equivariant map from the universal cover of M to the tropical projective space (see theorem 133). For Teichmüller spaces it is known that this equivariant map induces a measured lamination on the surface. This work can possibly lead to the discovery of analogous structures for the general case. For example an action of $\pi_1(M)$ on a tropical projective space induces a degenerate metric on the surface, and this metric can be used to associate a length with each curve. Anyway it is not clear up to know how to classify these induced structures. See chapter 7 for details.

Chapter 1

Character varieties

Let Γ be a finitely generated group and $\mathbb{K}$ a field of characteristic 0. In this chapter we describe the set of representations of Γ in $GL_n(\mathbb{K})$, denoted by $\mathrm{Hom}(\Gamma, GL_n(\mathbb{K}))$, and its behavior under the action by conjugation of $PGL_n(\mathbb{K})$. The set $\mathrm{Hom}(\Gamma, GL_n(\mathbb{K}))$ has a natural structure of affine algebraic set, well defined up to polynomial maps, and for this reason it will be called the variety of representations. The set $\mathrm{Hom}(\Gamma, GL_n(\mathbb{K}))/PGL_n(\mathbb{K})$ does not have such a structure in general. Instead of considering the quotient for the action of $PGL_n(\mathbb{K})$, it is convenient to consider the set of characters of representations in $\mathrm{Hom}(\Gamma, GL_n(\mathbb{K}))$.

If $\mathbb{K}$ is algebraically closed, the general methods of the geometric invariant theory, and some results about matrices invariants, make it possible to construct an affine algebraic set $\mathrm{Char}(\Gamma, GL_n(\mathbb{K}))$, with a map $t : \mathrm{Hom}(\Gamma, GL_n(\mathbb{K})) \longrightarrow \mathrm{Char}(\Gamma, GL_n(\mathbb{K}))$ in such a way that $\mathrm{Char}(\Gamma, GL_n(\mathbb{K}))$ is in natural bijection with the set of characters of representations in $\mathrm{Hom}(\Gamma, GL_n(\mathbb{K}))$, and the map t associate to every representation its character. Absolutely irreducible representations have the same characters if and only if they are conjugated, hence the affine algebraic set $\mathrm{Char}(\Gamma, GL_n(\mathbb{K}))$ plays the role of the quotient $\mathrm{Hom}(\Gamma, GL_n(\mathbb{K}))/PGL_n(\mathbb{K})$, at least for irreducible representations.

We need to extend these results for the case of real closed fields. If $\mathbb{F}$ is a real closed field, the affine algebraic set $\mathrm{Char}(\Gamma, GL_n(\mathbb{F}))$ is not in bijection with the set of characters of representations in $\mathrm{Hom}(\Gamma, GL_n(\mathbb{K}))$, as in this case the map t is not, in general, surjective. Anyway we can show that in this case the set of characters is a closed semi-algebraic subset of $\mathrm{Char}(\Gamma, GL_n(\mathbb{F}))$, that we will denote by $\overline{\mathrm{Char}}(\Gamma, GL_n(\mathbb{F}))$, and that the map $t : \mathrm{Hom}(\Gamma, GL_n(\mathbb{K})) \longrightarrow \overline{\mathrm{Char}}(\Gamma, GL_n(\mathbb{F}))$ sends closed invariant subsets in closed sets, and open invariant subsets in open sets (see theorem 11 and corollary 12).

1.1 Algebraic groups

We give some definitions about groups that are needed to read the sections 1.3 and 1.4. For the whole chapter k will be a field of characteristic 0, and $\mathbb{K} \supset k$ an over-field.

1.1.1 Algebraic schemes

A k-**algebraic scheme** is a scheme X with a morphism of finite type $X \longrightarrow \mathrm{Spec}(k)$. If X, Y are k-algebraic schemes, a k-**morphism** $X \longrightarrow Y$ is a morphism commuting with the given morphisms to $\mathrm{Spec}(k)$.

If X is a k-algebraic scheme, a natural extension is defined by $X \times_k \mathbb{K} \longrightarrow \mathbb{K}$ that is a $\mathbb{K}$-algebraic scheme. A $\mathbb{K}$-**valued point** of X is a morphism $\mathrm{Spec}(\mathbb{K}) \longrightarrow X$. If $\mathbb{K}$ is algebraically closed, such a point is called a **geometric point**. The set of all $\mathbb{K}$-valued points of X is denoted by $X(\mathbb{K})$.

If A is a finitely generated k-algebra, the scheme $X = \mathrm{Spec}(A)$ is an **affine** k-**algebraic scheme**. If $a_1, \ldots, a_n$ generate A as a k-algebra, there is a unique morphism $\phi : k[x_1, \ldots, x_n] \longrightarrow A$ such that $\phi(x_i) = a_i$. Let $I = \ker(\phi)$. Then there is a natural bijection between $X(\mathbb{K})$ and the **affine algebraic set** $\{x \in \mathbb{K}^n \mid \forall f \in I : f(x) = 0\}$.

Vice versa, if V is an affine algebraic subset of $\mathbb{K}^n$ defined over k, we denote by $\mathcal{A}_V$ its ring of coordinates, a k-algebra. Then if $X = \mathrm{Spec}(\mathcal{A}_V)$, there is a natural bijection between V and $X(\mathbb{K})$.

1.1.2 Algebraic groups

A k-**algebraic group** is given by a smooth k-algebraic scheme G and k-morphisms $\mu : G \times_k G \longrightarrow G$, $\beta : G \longrightarrow G$, $e : \mathrm{Spec}(k) \longrightarrow G$ satisfying the usual identities. If G is a k-algebraic group, then the set $G(\mathbb{K})$, with the induced operations, is a group.

A k-**algebraic subgroup** of G is a closed k-algebraic subscheme of G that is preserved by the operations of G. A k-**homomorphism** of k-algebraic groups $\phi : G \longrightarrow H$ is a k-morphism preserving the group operations. Then $\ker\phi$ is a normal k-algebraic subgroup of G, and $\mathrm{Im}\,\phi$ is a k-algebraic subgroup of H. If G and H are k-algebraic groups, then their direct product $G \times_k H$ has a canonical k-algebraic group structure.

If a k-algebraic group X is reduced, then it is smooth. A k-algebraic group is connected (for the Zariski topology) if and only if it is irreducible. There is a unique irreducible component G_0 of G containing the neutral element. It is called the **identity component**, and is a closed normal subgroup of finite index.

1.1.3 Examples

For example, the **additive group** is defined as $G_a = \mathrm{Spec}(\mathbb{Q}[x])$, the affine line, with group operation inherited by $(\mathbb{Q}, +)$. The set of $\mathbb{K}$-points $G_a(\mathbb{K})$ is in bijection with $\mathbb{K}$ an affine algebraic set, with group structure $(\mathbb{K}, +)$.

The **multiplicative group** is defined as $G_m = \mathrm{Spec}(\mathbb{Q}[x, y]/(xy - 1))$, with group operation inherited by $(\mathbb{Q}, \cdot)$. By definition, the set of $\mathbb{K}$-points $G_m(\mathbb{K})$ is in bijection with $\{(x, y) \in \mathbb{K}^2 \mid xy = 1\}$, an affine algebraic set. The projection on the first coordinate identifies it with $\mathbb{K}^*$, with group structure $(\mathbb{K}^*, \cdot)$. A k-**algebraic torus** is a k-algebraic group that is isomorphic to a group of the form $(G_m \times_{\mathbb{Q}} \mathrm{Spec}(k))^n$.

The most important examples for us are the groups of matrices. Let $\mathcal{A}(GL_n) = \mathbb{Q}[(a_{i,j})_{i,j\in\{1\dots n\}}, b]/\left(b\det(a_{i,j}) - 1\right)$, and $GL_n = \mathrm{Spec}(\mathcal{A}(GL_n))$, an affine $\mathbb{Q}$-algebraic scheme, with the standard group operations. Let $M_n(\mathbb{K})$ denote the set of all $n \times n$ matrices with entries in $\mathbb{K}$, an affine space of dimension n^2. The set of $\mathbb{K}$-points $GL_n(\mathbb{K})$ is in bijection with the **general linear group** of $\mathbb{K}$, an affine algebraic set with a group structure. By our definition, $GL_n(\mathbb{K}) \subset \{(a_{i,j}, b) \in M_n(\mathbb{K}) \times \mathbb{K} \mid b\det(a_{i,j}) = 1\}$ an affine algebraic set. The projection on $M_n(\mathbb{K})$ gives the usual immersion

$$GL_n(\mathbb{K}) = \{(a_{i,j}) \in M_n(\mathbb{K}) \mid \det(a_{i,j}) \neq 0\}$$

Similarly, let $\mathcal{A}(SL_n) = \mathbb{Q}[(a_{i,j})_{i,j\in\{1\dots n\}}]/\left(\det(a_{i,j}) - 1\right)$, and $SL_n = \mathrm{Spec}(\mathcal{A}(SL_n))$, an affine $\mathbb{Q}$-algebraic scheme, with the standard group operations. Then the set of $\mathbb{K}$-points $SL_n(\mathbb{K})$ is in bijection with the **special linear group** of $\mathbb{K}$, an affine algebraic set with a group structure:

$$SL_n(\mathbb{K}) = \{(a_{i,j}) \in M_n(\mathbb{K}) \mid \det(a_{i,j}) = 1\}$$

Let $\mathcal{A}(SL_n^{\pm}) = \mathbb{Q}[(a_{i,j})_{i,j\in\{1\dots n\}}]/\left((\det(a_{i,j}) - 1)(\det(a_{i,j}) + 1)\right)$, and $SL_n^{\pm} = \mathrm{Spec}(\mathcal{A}(SL_n^{\pm}))$, an affine $\mathbb{Q}$-algebraic scheme, with the standard group operations. Then the set of $\mathbb{K}$-points $SL_n^{\pm}(\mathbb{K})$ is in bijection with the group of matrices with determinant 1 or -1 with entries in $\mathbb{K}$, an affine algebraic set with a group structure:

$$SL_n^{\pm}(\mathbb{K}) = \{(a_{i,j}) \in M_n(\mathbb{K}) \mid \det(a_{i,j}) = 1\}$$

A **representation** of a k-algebraic group G is a k-homomorphism $G \longrightarrow GL_n \times_{\mathbb{Q}} \mathrm{Spec}(k)$.

The groups SL_n and $SL_n^{\pm}$ are $\mathbb{Q}$-algebraic normal subgroups of GL_n. A **linear k-algebraic group** is a k-algebraic subgroup of $GL_n \times_{\mathbb{Q}} \mathrm{Spec}(k)$. An **affine k-algebraic group** is a k-algebraic group that is an affine scheme. A theorem states that every affine k-algebraic group is isomorphic to a **linear k-algebraic group**.

An algebraic group is **reductive** if its radical is a torus, and it is **linearly reductive** if every representation of G is completely reducible. As we are in characteristic 0, G is reductive if and only if it is linearly reductive. In this case, $G_0' = [G_0, G_0]$ is semisimple, and there exists a torus T such that G_0 is isogenous to $T \times G_0'$. Every semisimple algebraic group and every algebraic torus is reductive, as is every general linear group.

1.1.4 Actions

A **left k-action** of a k-algebraic group G on a k-algebraic scheme X is given by a k-morphism $\sigma : G \times_k X \longrightarrow X$ satisfying the usual rules. For this subsection, let $\sigma : G \times_k X \longrightarrow X$ be an action.

Let $x : \mathrm{Spec}(\mathbb{K}) \longrightarrow X$ be a $\mathbb{K}$-valued point. We denote by $x^{\mathbb{K}} : \mathrm{Spec}(\mathbb{K}) \longrightarrow X \times_k \mathrm{Spec}(\mathbb{K})$ its lift to the extension.

The **orbit** of x is the image of the map

$$\psi_x : G \times_k \mathrm{Spec}(\mathbb{K}) \overset{(1_G \times x^{\mathbb{K}})}{\longrightarrow} G \times_k (X \times_k \mathrm{Spec}(\mathbb{K})) \overset{\sigma}{\longrightarrow} X \times_k \mathrm{Spec}(\mathbb{K})$$

and it is denoted by $O(x)$. An orbit is always locally closed.

Let $I = \mathrm{Im}(x^{\mathbb{K}})$, a subscheme of $X \times_k \mathrm{Spec}(\mathbb{K})$. The inverse image of I through the map ψ_x is a $\mathbb{K}$-algebraic subscheme of $G \times_k \mathrm{Spec}(\mathbb{K})$, denoted by $S(x)$. This is a $\mathbb{K}$-algebraic subgroup called the **stabilizer** of x.

The action is said to be **transitive** if there exists a k-valued point x such that $O(x) = X$. In this case all the orbits of k-valued points are equal to X, and X is said to be an **homogeneous space**. If an homogeneous space is reduced, then it is smooth.

Let G be a k-algebraic group, and H be a k-algebraic subgroup. A **quotient** of G by H is a pair $(G/H, a)$, where G/H is an homogeneous space for G, and $a \in G/H$ is a base point whose isotropy group is H, such that for every pair (X, b), where X is an homogeneous space and $b \in X$ is an element whose isotropy group contains H, there exists a unique G-equivariant morphism $\phi : G/H \longrightarrow X$ with $\phi(a) = b$.

A theorem states that a quotient always exists, and it is unique up to G-equivariant isomorphisms. Moreover if G is affine, the quotient is also affine. If H is a normal k-algebraic subgroup of G, then the quotient G/H inherits a structure of k-algebraic group, and it is called the **quotient group**. If the group G is affine, the quotient group is affine.

For example, let $G = GL_n$, and let H be the subgroup of scalar multiples of the identity. The subgroup H is a normal $\mathbb{Q}$-algebraic subgroup, hence the quotient $PGL_n = GL_n/H$ is an affine $\mathbb{Q}$-algebraic group, called the **projective general linear group**. Similarly, if $G = SL_n$, the quotient $PSL_n = SL_n/(H \cap SL_n)$ is an affine $\mathbb{Q}$-algebraic group called the **projective special linear group**.

1.1.5 Properties of the linear group

We denote by $\pi : GL_n \longrightarrow PGL_n$ and $\pi : SL_n \longrightarrow PSL_n$ the quotient maps. If $\mathbb{K}$ is algebraically closed, then $\pi_{|SL_n(\mathbb{K})} : SL_n(\mathbb{K}) \longrightarrow PGL_n(\mathbb{K})$ is surjective and its kernel is the group $\mathbb{Z}_n$ of n-th roots of unity of $\mathbb{K}$. In particular $PGL_n(\mathbb{K}) = PSL_n(\mathbb{K})$. If $\mathbb{K}$ is real closed, the behavior depends on the parity of n. If n is odd, then $\pi_{|SL_n(\mathbb{K})} : SL_n(\mathbb{K}) \longrightarrow PGL_n(\mathbb{K})$ is a bijection, and it identifies $PGL_n(\mathbb{K}) = PSL_n(\mathbb{K}) = SL_n(\mathbb{K})$. If n is even, then $PSL_n(\mathbb{K})$ is a subgroup of index 2 in $PGL_n(\mathbb{K})$, and the kernel of $\pi_{|SL_n(\mathbb{K})} : SL_n(\mathbb{K}) \longrightarrow PSL_n(\mathbb{K})$ is $\{\pm \mathrm{Id}\}$. To unify the statements, consider the map $\pi_{|SL_n^{\pm}(\mathbb{K})} : SL_n^{\pm}(\mathbb{K}) \longrightarrow PGL_n(\mathbb{K})$. In both cases this map is surjective, with kernel $\{\pm \mathrm{Id}\}$.

The group PGL_n acts by conjugation on GL_n, SL_n, $SL_n^{\pm}$, PGL_n and PSL_n. Given a field $\mathbb{K}$, the action by conjugation of $PGL_n(\mathbb{K})$ extends to $M_n(\mathbb{K})$: for $A \in PGL_n(\mathbb{K})$, let $\bar{A}$ be a lift of A in $GL_n(\mathbb{K})$, the map

$$\mathrm{Ad}(A) : M_n(\mathbb{K}) \ni G \longrightarrow \bar{A}^{-1} G \bar{A} \in M_n(\mathbb{K})$$

is a $\mathbb{K}$-algebra isomorphism of $M_n(\mathbb{K})$. This induces a map

$$\mathrm{Ad} : PGL_n(\mathbb{K}) \longrightarrow \mathrm{Aut}(M_n(\mathbb{K}))$$

The map Ad is injective because the center of $GL_n(\mathbb{K})$ is exactly the set of multiples of the identity, and it is surjective by the Skolem-Noether theorem.

The fundamental invariant for this action is the trace function:

$$\mathrm{tr} : M_n(\mathbb{K}) \longrightarrow \mathbb{K}$$

The trace defines a symmetric non-degenerate bilinear form:

$$M_n(\mathbb{K}) \times M_n(\mathbb{K}) \ni (A, B) \longrightarrow \mathrm{tr}(AB) \in \mathbb{K}$$

Hence if $A_1, \ldots, A_{n^2}$ is a basis of $M_n(\mathbb{K})$, every matrix B is determined by the values $\mathrm{tr}(BA_i)$.

1.2 Representation varieties

1.2.1 Representations

Let Γ be a group. A **representation** of Γ is a group homomorphism $\rho : \Gamma \longrightarrow GL_n(\mathbb{K})$ or $\rho : \Gamma \longrightarrow PGL_n(\mathbb{K})$.

A subspace $H \subset \mathbb{K}^n$ ($H \subset \mathbb{KP}^{n-1}$ for projective representations) is **invariant** for a representation ρ if

$$\forall \gamma \in \Gamma : \rho(\gamma)(H) \subset H$$

A representation ρ is **irreducible** if the only invariant subspaces are (0) and $\mathbb{K}^n$ (if the only invariant subspace is $\mathbb{KP}^{n-1}$, for projective representations), else it is **reducible**. It is **absolutely irreducible** if it is irreducible as a representation in $GL_n(\mathbb{F})$ (or $PGL_n(\mathbb{F})$) where $\mathbb{F}$ is the algebraic closure of $\mathbb{K}$, else it is **absolutely reducible**.

An equivalent characterization of absolutely irreducible representations is provided by the Burnside lemma:

Lemma 1. (Burnside) *A representation ρ in $GL_n(\mathbb{K})$ (or $PGL_n(\mathbb{K})$) is absolutely irreducible if and only if the image $\rho(\Gamma)$ (or the inverse image $\pi^{-1}(\rho(\Gamma))$, for projective representations) spans $M_n(\mathbb{K})$ as a $\mathbb{K}$-vector space.*

Proof : See [Ba80, Lemma 1.2]. □

Two representations ρ, ρ' are **conjugated** if there exists $a \in PGL_n(\mathbb{K})$ such that for all $\gamma \in \Gamma$ we have

$$\rho(\gamma) = Ad(a)(\rho'(\gamma))$$

In this case we will write $\rho \sim \rho'$. The **character** of a representation ρ is the function

$$\chi_\rho : \Gamma \ni \gamma \longrightarrow \operatorname{tr}(\rho(\gamma)) \in \mathbb{K}$$

By the conjugation-invariance of the trace, two conjugated representations have the same character. The converse holds for irreducible representations:

Proposition 2. *Let ρ, ρ' be two representations, and suppose that ρ is absolutely irreducible. Then*

$$\rho \sim \rho' \Leftrightarrow \chi_\rho = \chi_{\rho'}$$

Proof : (See also [Na00, Thm. 6.12] for a more general statement). By the Burnside lemma there exist elements $\gamma_1, \dots, \gamma_{n^2} \in \Gamma$ such that $\rho(\gamma_1), \dots, \rho(\gamma_{n^2})$ forms a basis of $M_n(\mathbb{K})$. Let ϕ be the $\mathbb{K}$-linear map sending $\rho(\gamma_i)$ in $\rho'(\gamma_i)$. As $\operatorname{tr}(\rho(\gamma_i)\rho(\gamma_j)) = \operatorname{tr}(\rho(\gamma_i\gamma_j)) = \operatorname{tr}(\rho'(\gamma_i\gamma_j)) = \operatorname{tr}(\rho'(\gamma_i)\rho'(\gamma_j))$, the map ϕ is orthogonal with reference to the bilinear form defined by the trace, and, in particular, it is bijective.

First note that for every $\gamma \in \Gamma$, we have $\phi(\rho(\gamma)) = \rho'(\gamma)$: we have that $\operatorname{tr}(\phi(\rho(\gamma))\rho'(\gamma_i)) = \operatorname{tr}(\phi(\rho(\gamma))\phi(\rho(\gamma_i))) = \operatorname{tr}(\rho(\gamma)\rho(\gamma_i)) = \operatorname{tr}(\rho'(\gamma)\rho'(\gamma_i))$.

We want to prove that ϕ is a $\mathbb{K}$-algebra isomorphism. Given $A, B \in M_n(\mathbb{K})$, we need to prove that $\phi(AB) = \phi(A)\phi(B)$. We can write

$$A = \sum_{i=1}^{n^2} a_i \rho(\gamma_i) \qquad B = \sum_{i=1}^{n^2} b_i \rho(\gamma_i)$$

Then

$$AB = \sum_{i,j=1}^{n^2} a_i b_j \rho(\gamma_i\gamma_j)$$

We conclude because $\phi(A) = \sum_{i=1}^{n^2} a_i\rho'(\gamma_i)$, $\phi(B) = \sum_{i=1}^{n^2} b_i\rho'(\gamma_i)$, and $\phi(AB) = \sum_{i,j=1}^{n^2} a_i b_j \rho'(\gamma_i\gamma_j)$.

Hence ϕ is a $\mathbb{K}$-algebra isomorphism sending $\rho(\gamma)$ in $\rho'(\gamma)$. By the Skolem-Noether theorem, ϕ is the conjugation with an element of $PGL_n(\mathbb{K})$. $\square$

1.2.2 Algebraic structure

In the following Γ is assumed to be a finitely generated group and $k \subset \mathbb{K}$, as usual, fields of characteristic 0. Let G be a k-algebraic group. The set of all group homomorphisms from Γ in $G(\mathbb{K})$ will be denoted by $\mathrm{Hom}(\Gamma, G(\mathbb{K}))$.

A **presentation** of Γ is a surjective homomorphism $P : \mathbb{Z}^{(m)} \longrightarrow \Gamma$, where $\mathbb{Z}^{(m)}$ is the free group with m generators $z_1, \ldots, z_m$. Given a presentation P, its **generators** are the elements $\gamma_i = P(z_i) \in \Gamma$, and its **relations** are the elements of $\ker(P) \subset \mathbb{Z}^{(m)}$.

Given a presentation P, the map

$$H_P : \mathrm{Hom}(\Gamma, G(\mathbb{K})) \ni \rho \longrightarrow (\rho(\gamma_1), \ldots, \rho(\gamma_m)) \in G(\mathbb{K})^m$$

identifies $\mathrm{Hom}(\Gamma, G(\mathbb{K}))$ with the subset of all m-tuples of elements of $G(\mathbb{K})$ satisfying the relations of P.

Each of these relations states that a certain word in the generators $\gamma_1, \ldots, \gamma_m$ is equal to the identity, hence as the product $G \times G \longrightarrow G$ is a morphism, the set of m-tuples in $G(\mathbb{K})^m$ satisfying the relation is closed. Arbitrary intersection of closed is closed, hence the image of H_P is a closed subset of $G(\mathbb{K})^m$.

If $Q : \mathbb{Z}^{(s)} \longrightarrow \Gamma$ is another presentation, with generators $\delta_1, \ldots, \delta_s$, the map $c = H_Q \circ {H_P}^{-1}$ can be written explicitly. For every i we choose a $\zeta_i \in P^{-1}(\delta_i) \subset \mathbb{Z}^{(m)}$. Every ζ_i is a word

$$\zeta_i = z_{j_{i1}}^{s_{i1}} z_{j_{i2}}^{s_{i2}} \cdots z_{j_{ik_i}}^{s_{ik_i}}$$

for suitable multi-indices j_i e s_i. The map c can be written as:

$$G(\mathbb{K})^m \ni (g_1 \ldots g_m) \longrightarrow (g_{j_{11}}^{s_{11}} g_{j_{12}}^{s_{12}} \cdots g_{j_{1k_1}}^{s_{1k_1}}, \ldots) \in G(\mathbb{K})^s$$

And this map is an invertible morphism, identifying the images of H_P and H_Q.

The equations defining the image of H_P depends only on P, and not on the specific field $\mathbb{K}$. Hence we can actually define a k-algebraic scheme $H = \mathrm{Hom}(\Gamma, G)$, such that for every field $\mathbb{K}$ we have $H(\mathbb{K}) = \mathrm{Hom}(\Gamma, G(\mathbb{K}))$.

If G is an affine k-algebraic group, the scheme $\mathrm{Hom}(\Gamma, G)$ is an affine k-algebraic scheme, hence $\mathrm{Hom}(\Gamma, G(\mathbb{K}))$ can be embedded in an affine space $\mathbb{K}^M$ as an affine algebraic set. In this case G is isomorphic to a linear algebraic group, hence the elements of $\mathrm{Hom}(\Gamma, G(\mathbb{K}))$ are representations of

Γ. For this reason, the scheme $\mathrm{Hom}(\Gamma, G)$ will be called **representation scheme**, and the affine algebraic set $\mathrm{Hom}(\Gamma, G(\mathbb{K}))$ will be called **representation variety**.

A group homomorphism $h : \Gamma \longrightarrow \Delta$ induces a map:

$$h^* : \mathrm{Hom}(\Delta, G(\mathbb{K})) \ni \rho \longrightarrow \rho \circ h \in \mathrm{Hom}(\Gamma, G(\mathbb{K}))$$

This map comes from a morphism $h^* : \mathrm{Hom}(\Delta, G) \longrightarrow \in \mathrm{Hom}(\Gamma, G)$. The image of h^* is closed. If h is surjective, the map h^* is injective. Even if h is injective, the map h^* is not necessarily surjective.

A k-algebraic group homomorphism $h : G \longrightarrow H$ induces a map:

$$h_* : \mathrm{Hom}(\Gamma, G(\mathbb{K})) \ni \rho \longrightarrow h \circ \rho \in \mathrm{Hom}(\Gamma, H(\mathbb{K}))$$

This map comes from a morphism $h_* : \mathrm{Hom}(\Gamma, G) \longrightarrow \in \mathrm{Hom}(\Gamma, H)$. The image of h_* is closed. If h is injective, the map h_* is injective. If h is bijective, the map h_* is bijective. Even if h is surjective, the map h_* is not necessarily surjective. If Γ is a free group, and $h : G \longrightarrow H$ is surjective, then every representation $\rho : \Gamma \longrightarrow H$ can be lifted to a representation in G, hence in this case the map h_* is surjective.

Let Δ be a finitely generated group, and $h : \mathbb{Z}^{(m)} \longrightarrow \Delta$ a presentation. Suppose G, H are algebraic groups with a surjective morphism $k : G \longrightarrow H$. Then $h^* : \mathrm{Hom}(\Delta, H) \ni \rho \longrightarrow \rho \circ h \in \mathrm{Hom}(\mathbb{Z}^{(m)}, H)$ identifies $\mathrm{Hom}(\Delta, H)$ with a closed subscheme $Y \subset \mathrm{Hom}(\mathbb{Z}^{(m)}, H)$, and the inverse image $X = (k_*)^{-1}(Y)$ through the map $k_* : \mathrm{Hom}(\mathbb{Z}^{(m)}, G) \longrightarrow \mathrm{Hom}(\mathbb{Z}^{(m)}, H)$ is a closed subscheme of $\mathrm{Hom}(\mathbb{Z}^{(m)}, G)$ with a canonical surjective map $X \longrightarrow \mathrm{Hom}(\Delta, H)$.

Let H be a k-algebraic group acting on a $\mathbb{K}$-algebraic group G. Then the functor $h \longrightarrow h_*$, induces a canonical action of H on $\mathrm{Hom}(\Gamma, G)$. For example, if $G = GL_n, SL_n, SL_n^{\pm}, PGL_n$ or $, PSL_n$, the action of PGL_n by conjugation on G induces an action of PGL_n on $\mathrm{Hom}(\Gamma, G)$.

1.2.3 Trace functions

For this section, let $G = GL_n, SL_n$ or $SL_n^{\pm}$, an affine algebraic group. Every $\gamma \in \Gamma$ defines a polynomial function

$$\tau_\gamma : \mathrm{Hom}(\Gamma, G(\mathbb{K})) \ni \rho \longrightarrow \chi_\rho(\gamma) \in \mathbb{K}$$

these functions comes from functions $\tau_\gamma \in \mathcal{A}(\mathrm{Hom}(\Gamma, G))$ and they will be called **trace functions**.

We denote by $\mathrm{Hom}(\Gamma, G(\mathbb{K}))_{a.r.r.}$ the subset of all absolutely reducible representations. This subset is a closed algebraic subset, as in [Na00]. Let $\gamma_1, \ldots, \gamma_{n^2}$ be elements of Γ. Given a representation ρ, the elements $\rho(\gamma_i)$

form a basis of $M_n(\mathbb{K})$ if and only if the matrix $(\mathrm{tr}(\rho(\gamma_i\gamma_j)))$ is non-singular. With $\{\gamma_i\}$ we associate a discriminant function:

$$\Delta_{\gamma_1,\ldots,\gamma_{n^2}}(\rho) = \det\left(\tau_{\gamma_i\gamma_j}(\rho)\right)$$

By the Burnside lemma, a representation is absolutely reducible if and only if all discriminant functions are zero:

$$\mathrm{Hom}(\Gamma, G(\mathbb{K}))_{a.r.r.} = \{\rho \mid \forall\gamma_1,\ldots,\gamma_{n^2} \in \Gamma : \Delta_{\gamma_1,\ldots,\gamma_{n^2}}(\rho) = 0\}$$

This defines a closed subscheme $\mathrm{Hom}(\Gamma, G)_{a.r.r.}$. By the Hilbert basis theorem, only a finite number of these equations are required. Hence there exists a finite number of elements $\gamma_1,\ldots,\gamma_s \in \Gamma$ such that for every absolutely irreducible representation ρ, the matrices $\{\rho(\gamma_i)\}$ spans $M_n(\mathbb{K})$ as a $\mathbb{K}$-vector space. We define also $\mathrm{Hom}(\Gamma, G)_{a.i.r}$ as the complement of $\mathrm{Hom}(\Gamma, G)_{a.r.r.}$, the set of absolutely irreducible representations, that is a Zariski open subset.

Let $V \subset \mathrm{Hom}(\Gamma, G(\mathbb{K}))$ be an irreducible algebraic subset. We denote by $\mathcal{A}(V)$ its ring of coordinates, and by $\mathbb{K}(V)$ its field of fractions. A point $\rho \in V$ is a representation $\rho : \Gamma \longrightarrow G(\mathbb{K})$. We write $\rho(\gamma) = \left(a^\gamma_{i,j}(\rho)\right)$, where $a^\gamma_{i,j} \in \mathcal{A}(V)$ as they are restriction of polynomial functions in $\mathrm{Hom}(\Gamma, G(\mathbb{K}))$. Note that for every $\rho \in V$, $\det\left(a^\gamma_{i,j}(\rho)\right) \neq 0$, hence the element $\det\left(a^\gamma_{i,j}\right)$ is invertible in $\mathbb{K}(V)$. The map:

$$\Gamma \ni \gamma \longrightarrow \left(a^\gamma_{i,j}\right) \in GL_n(\mathbb{K}(V))$$

is the canonical representation in $G(\mathbb{K}(V))$, that will be denoted by $\mathcal{R}_V$. The character of $\mathcal{R}_V$ is the function $\chi_{\mathcal{R}_V}(\gamma) = \sum_i a^\gamma_{i,i} = \tau_\gamma$.

Consider the action by conjugation of $PGL_n(\mathbb{K})$ on $\mathrm{Hom}(\Gamma, G(\mathbb{K}))$.

Proposition 3. *Let $V \subset \mathrm{Hom}(\Gamma, G(\mathbb{K}))$ be an irreducible component. Then V is invariant for the action of $PGL_n(\mathbb{K})$ by conjugation.*

Proof : The same statement for $n = 2$ is proved in [CS83, prop. 1.1.1]. Consider the set $V \times PGL_n(\mathbb{K})$, this is the product of two irreducible affine algebraic sets, hence it is an irreducible affine algebraic set. The map

$$f : V \times PGL_n(\mathbb{K}) \ni (\rho, A) \longrightarrow \mathrm{Ad}(A)(\rho) \in \mathrm{Hom}(\Gamma, G(\mathbb{K}))$$

is a regular map, hence the Zariski closure of the image $f(V \times PGL_n(\mathbb{K}))$ is an irreducible affine algebraic set, hence it is contained in an irreducible component of $\mathrm{Hom}(\Gamma, G(\mathbb{K}))$. But as $V = f(V \times \{\mathrm{Id}\})$, then $f(V \times PGL_n(\mathbb{K})) \subset V$. □

1.3 Geometric invariant theory

In this section we review some basic facts of geometric invariant theory, see [MFK94] for details. In the following k will be a field of characteristic 0, $\mathbb{K} \supset k$ an over-field. G will be a k-algebraic group acting on a k-algebraic scheme X.

1.3.1 Definition of quotients

We will denote by $1_X : X \longrightarrow X$ the identity morphism, p_1 and p_2 are the projections from $X \times_k Y$ to X and Y. If $f : X \longrightarrow Y_1$ and $g : X \longrightarrow Y_2$, then they induce $(f, g) : X \longrightarrow Y_1 \times_k Y_2$, and if $f : X_1 \longrightarrow Y_1$ and $g : X_2 \longrightarrow Y_2$, then their product is denoted by $f \times g : X_1 \times_k X_2 \longrightarrow Y_1 \times_k Y_2$. The map $\Delta : X \longrightarrow X \times_k X$ is the diagonal map.

The action σ is said to be

1. **closed** if for all geometric points x of X over some over-field $\mathbb{K} \supset k$, the orbit $O(x) \subset X \times_k \mathrm{Spec}(\mathbb{K})$ is closed.
2. **separated** if the image of $(\sigma, p_2) : G \times_k X \longrightarrow X \times_k X$ is closed.
3. **proper** if (σ, p_2) is proper.
4. **free** if (σ, p_2) is a closed immersion.

Note that a free action is also separated and proper.

Let (Y, ϕ) be a pair consisting of a pre-scheme Y over k and a k-morphism $\phi : X \longrightarrow Y$. The pair (Y, ϕ) is called a **sub-quotient** if the two maps

$$\phi \circ \sigma, \phi \circ p_2 : G \times_k X \longrightarrow Y$$

are equal. Intuitively speaking, we ask that points in X in the same G-orbit go in the same point of Y.

The pair (Y, ϕ) is called a **categorical quotient** if it is a sub-quotient and if for every other sub-quotient (Z, ψ) there is a unique k-morphism $\chi : Y \longrightarrow Z$ such that $\psi = \chi \circ \phi$. A categorical quotient, it it exists, is unique up to isomorphisms. Let (Y, ϕ) be a categorical quotient. By the universal mapping property, if X is reduced, or connected, or irreducible, or locally integral, or locally integral and normal, then also Y is.

The pair (Y, ϕ) is a **semi-geometric quotient** if

1. (Y, ϕ) is a sub-quotient.
2. ϕ is surjective.
3. The fundamental sheaf $\mathcal{O}_Y$ is the subsheaf of $\phi_*(\mathcal{O}_X)$ consisting of invariant functions, i.e. if $f \in \Gamma(U, \phi_*(\mathcal{O}_X)) = \Gamma(\phi^{-1}(U), \mathcal{O}_X)$, then

$f \in \Gamma(U, \mathcal{O}_Y)$ if and only if, denoted by $F : \phi^{-1}(U) \longrightarrow A^1$ the morphism defined by f, the two maps

$$F \circ \sigma, F \circ p_2 : G \times \phi^{-1}(U) \longrightarrow A^1$$

are equal.

The pair (Y, ϕ) is a **universal categorical quotient** if for every morphism $Y' \longrightarrow Y$, denoted $X' = X \times_Y Y'$ and $\phi' : X' \longrightarrow Y'$, then (Y', ϕ') is a categorical quotient of X' by G. Every universal categorical quotient is a semi-geometric quotient.

The pair (Y, ϕ) is called a **geometric quotient** if:

1. (Y, ϕ) is a semi-geometric quotient.
2. The image of the map $(\sigma, p_2) : G \times_k X \longrightarrow X \times_k X$, is $X \times_Y X$.
3. ϕ is submersive (i.e. a subset $U \subset Y$ is open if and only if $\phi^{-1}(U)$ is open in X).

If a pair (Y, ϕ) is a geometric quotient, then it is also a categorical quotient, hence it is also unique. If Y is a k-algebraic scheme and ϕ is of finite type, then the condition on the image of the map (σ, p_2) is equivalent to the following condition: the geometric fibers of ϕ are precisely the orbits of the geometric points of X, over an algebraically closed over-field $\mathbb{K} \supset k$. If X is normal and (Y, ϕ) is a geometric quotient, then Y is a k-algebraic scheme.

If a geometric quotient (Y, ϕ) exists, then the action is closed. The prescheme Y is a scheme if and only if the action is separated. In this case, Y is a k-algebraic scheme.

The pair (Y, ϕ) is a **universal geometric quotient** if for every morphism $Y' \longrightarrow Y$, denoted $X' = X \times_Y Y'$ and $\phi' : X' \longrightarrow Y'$, then (Y', ϕ') is a geometric quotient for of X' by G.

1.3.2 Actions of reductive groups

Proposition 4. *Suppose that the pair (Y, ϕ) has the following properties:*

1. *it is a sub-quotient.*
2. *$\mathcal{O}_Y$ is the sheaf of invariants of $\phi_*(\mathcal{O}_X)$.*
3. *if W is an invariant closed subset of X, then $\phi(W)$ is closed in Y.*
4. *if $\{W_i\}_{i \in I}$ is a set of invariant closed subsets of X, then $\phi(\bigcap_{i \in I} W_i) = \bigcap_{i \in I} \phi(W_i)$*

Then (Y, ϕ) is a categorical quotient, and ϕ is dominating and submersive.

Proof : See [MFK94, Chap. 0, rem. 6]. □

Suppose that G is reductive, k has characteristic 0, and that X is a k-algebraic affine scheme, i.e. $X = \mathrm{Spec}(R)$, where R is a finitely generated k-algebra. The group G acts dually on R (see [MFK94, Chap. 1, def. 1.2-1.3]). Let R_0 be the subring of invariant functions, $Y = \mathrm{Spec}(R_0)$, and $\phi : X \longrightarrow Y$ be the morphism induced by the inclusion $R_0 \longrightarrow R$.

Proposition 5. *The pair (Y, ϕ) satisfy the hypotheses of the previous proposition. Consequently:*

1. *R_0 is a finitely generated k-algebra, hence Y is a k-algebraic affine scheme.*
2. *(Y, ϕ) is a universal categorical quotient, hence it is a semi-geometric quotient, and the map ϕ is universally submersive.*
3. *If W_1 and W_2 are two closed disjoint invariant subsets of X, there exists an invariant function which is 0 on W_1 and 1 on W_2.*
4. *If the action of G on X is closed, then (Y, ϕ) is a universal geometric quotient.*

Proof : See [MFK94, Chap. 1, thm. 1.1]. □

Proposition 6. *Let U be an open invariant subscheme of X. The image of $\phi_{|U}$ is an open subscheme Y_U of Y. Then $(Y_U, \phi_{|U})$ is a categorical quotient for the action of G on U. The pair $(Y_U, \phi_{|U})$ is a geometric quotient for the action of G on U if and only if the geometric fibers of $\phi_{|U}$ are precisely the orbits of the geometric points of X, over an algebraically closed over-field $\mathbb{K} \supset k$.*

Proof : The image of $\phi_{|U}$ is open as ϕ is universally submersive. The pair $(Y_U, \phi_{|U})$ satisfies the hypotheses of the criterion, because (Y, ϕ) does, hence it is a categorical quotient for the action of G on U. □

Proposition 7. *Let W be a closed invariant subscheme of X. The image of $\phi_{|W}$ is a closed subscheme Y_W of Y. Then $(Y_W, \phi_{|W})$ is a categorical quotient for the action of G on W.*

Proof : The natural map $p : R \longrightarrow \mathcal{A}(W)$ is a surjective ring homomorphism, commuting with the dual action of G. Let $A_0 \subset \mathcal{A}(W)$ be the subring of invariant functions. Then, as G is reductive, $p(R_0) = A_0$ (see the discussion after [MFK94, Chap. 1, def. 1.5]). Hence $p^* : \mathrm{Spec}(A_0) \longrightarrow \mathrm{Spec}(R_0)$ is an immersion of $\mathrm{Spec}(A_0)$ with image equal to $\phi(W)$. □

Let G be a reductive k-algebraic group k-acting on a k-algebraic scheme X. A geometric point x of X is pre-stable with respect to the action if there exists an invariant affine open subset $U \subset X$ such that x is a point of U and the action of G on U is closed. The set of pre-stable geometric points is the set of geometric points of an open invariant subscheme (possibly empty) $X^s(\mathrm{Pre})$ of X.

Proposition 8. *The group G also acts on $X^s(Pre)$. Then there exists a universal geometric quotient (Y, ϕ) of $X^s(Pre)$ by G. Moreover ϕ is affine, and Y is a k-algebraic pre-scheme. Conversely, if $U \subset X$ is an invariant open set such that a geometric quotient (Z, ψ) exists and such that ψ is affine, then $U \subset X^s(Pre)$. Note that if G acts properly on U, and a geometric quotient (Z, ψ) exists, then ψ is affine.*

Proof : See [MFK94, Chap.1, prop. 1.9]. □

Proposition 9. *In the hypotheses of the above proposition, if, moreover, X is affine, the universal geometric quotient of $X^s(Pre)$ is a k-algebraic scheme. If (Y, ϕ) is the universal categorical quotient, every open invariant subset U of X such that $(Y_U, \phi_{|U})$ is a geometric quotient is contained in $X^s(Pre)$.*

Proof : Denoted $Y' = \phi(X^s(\mathrm{Pre}))$ and $\phi' = \phi_{X^s(\mathrm{Pre})}$, then (Y', ϕ') is a universal categorical quotient for the action of G on $X^s(\mathrm{Pre})$. As, by previous assertion, this action has a universal geometric quotient, it coincides with (Y', ϕ'). □

1.4 Character varieties

1.4.1 General construction

Let G be an affine k-algebraic group, and let H be a reductive k-algebraic group acting on G. Then H acts on $\mathrm{Hom}(\Gamma, G)$, again an affine k-algebraic scheme.

We denote by $A = \mathcal{A}(\mathrm{Hom}(\Gamma, G))$ the ring of coordinates of $\mathrm{Hom}(\Gamma, G)$, and by A_0 the subring of invariant functions. By proposition 5, the ring A_0 is finitely generated as a k-algebra. Let $C \subset A_0$ be a finite set of generators.

Let $\mathbb{K} \supset k$ be an algebraically closed field, consider the map

$$t : \mathrm{Hom}(\Gamma, G(\mathbb{K})) \ni \rho \longrightarrow f(\rho)_{f \in C} \in \mathbb{K}^{\mathrm{Card}(C)}$$

We will denote by $Q_H(\Gamma, G(\mathbb{K}))$ the Zariski closure of the image of this map, an affine k-algebraic set whose ring of coordinates is isomorphic to A_0.

The map t is dual to the inclusion map $A_0 \longrightarrow A$, hence it is identified with the semi-geometric quotient $\mathrm{Hom}(\Gamma, G) = \mathrm{Spec}(A) \longrightarrow \mathrm{Spec}\, A_0$ defined in proposition 5. As this semi-geometric quotient is surjective, the map $t : \mathrm{Hom}(\Gamma, G(\mathbb{K})) \longrightarrow Q_H(\Gamma, G(\mathbb{K}))$ is surjective. We set $Q_H(\Gamma, G) = \mathrm{Spec}(A_0)$.

If $C' \subset A_0$ is another finite set of generators, the pair $(Q_H(\Gamma, G(\mathbb{K})), t)$ defined by C' is isomorphic to the previous one, hence this construction does not depend on the choices.

Proposition 10. *The map $t : \mathrm{Hom}(\Gamma, G(\mathbb{K})) \longrightarrow Q_H(\Gamma, G(\mathbb{K}))$ has the following properties:*

1. *the map t is surjective.*

2. *if W is an invariant closed subset of* $\mathrm{Hom}(\Gamma, G(\mathbb{K}))$*, then $t(W)$ is closed in* $Q_H(\Gamma, G(\mathbb{K}))$*.*

3. *if $\{W_i\}_{i\in I}$ is a set of invariant closed subsets of* $\mathrm{Hom}(\Gamma, G(\mathbb{K}))$*, then* $t(\bigcap_{i\in I} W_i) = \bigcap_{i\in I} t(W_i)$*.*

4. *If* $W_1, W_2 \subset \mathrm{Hom}(\Gamma, G(\mathbb{K}))$ *are two closed disjoint invariant subsets, then there exists an invariant function which is* 0 *on* W_1 *and* 1 *on* W_2*.*

5. *t is submersive (i.e. a subset $U \subset Q_H(\Gamma, G(\mathbb{K}))$ is open if and only if $t^{-1}(U)$ is open in* $\mathrm{Hom}(\Gamma, G(\mathbb{K}))$*).*

Proof : If follows from proposition 5. □

Let $\mathrm{Hom}(\Gamma, G)^s(\mathrm{Pre})$ be the open subscheme of pre-stable points for the action. Its image will be denoted by $t(\mathrm{Hom}(\Gamma, G)^s(\mathrm{Pre})) = Q_H(\Gamma, G)^s(\mathrm{Pre})$, an open subscheme of $Q_H(\Gamma, G)$. By proposition 9, the map

$$t_{|\,\mathrm{Hom}(\Gamma,G)^s(\mathrm{Pre})} : \mathrm{Hom}(\Gamma, G)^s(\mathrm{Pre}) \longrightarrow Q_H(\Gamma, G)^s(\mathrm{Pre})$$

is a geometric quotient. Hence, there is a natural bijection between the set-theoretical quotient $\mathrm{Hom}(\Gamma, G(\mathbb{K}))^s(\mathrm{Pre})/H(\mathbb{K})$ and $Q_H(\Gamma, G(\mathbb{K}))^s(\mathrm{Pre})$.

1.4.2 Functorial properties

Let $h : \Gamma \longrightarrow \Delta$ be a group homomorphism. The image of the map $h^* : \mathrm{Hom}(\Delta, G) \longrightarrow \mathrm{Hom}(\Gamma, G)$ is an invariant closed subscheme, as h^* has closed image, and it is equivariant for the action of H. The composition map $t \circ h^* : \mathrm{Hom}(\Delta, G) \longrightarrow Q_H(\Gamma, G)$ has closed image, and it is a sub-quotient, hence there is a unique map

$$h^{\#} : Q_H(\Delta, G) \longrightarrow Q_H(\Gamma, G)$$

such that $t \circ h^* = h^{\#} \circ t : \mathrm{Hom}(\Delta, G) \longrightarrow Q_H(\Gamma, G)$. The map $h^{\#}$ has closed image.

If h is surjective, then h^* is injective, identifying $\mathrm{Hom}(\Delta, G)$ with its image. As $h^*(\mathrm{Hom}(\Delta, G))$ is a closed invariant subscheme, by proposition 7, $t : h^*(\mathrm{Hom}(\Delta, G)) \longrightarrow h^{\#}(Q_H(\Delta, G))$ is a semi-geometric quotient, hence there exists a unique map $k : h^{\#}(Q_H(\Delta, G)) \longrightarrow Q_H(\Delta, G)$ such that $k \circ t = t \circ (h^{\#})^{-1}$. This map is the inverse of $h^{\#}$, hence $h^{\#}$ is injective and it identifies $Q_H(\Delta, G)$ with its image $h^{\#}(Q_H(\Delta, G)) \subset Q_H(\Gamma, G)$.

Let G' be another affine k algebraic group with an action of H, and let $h : G \longrightarrow G'$ be a k-algebraic group homomorphism, equivariant with respect to the actions. The map $h_* : \mathrm{Hom}(\Gamma, G) \longrightarrow \in \mathrm{Hom}(\Gamma, G')$ has closed image, and as it is equivariant, its image is an invariant closed subscheme.

The composition map $t \circ h_* : \mathrm{Hom}(\Gamma, G) \longrightarrow Q_H(\Gamma, G')$ has closed image, and it is a sub-quotient, hence there is a unique map

$$h_\# : Q_H(\Gamma, G) \longrightarrow Q_H(\Gamma, G')$$

such that $t \circ h_* = h_\# \circ t : \mathrm{Hom}(\Gamma, G) \longrightarrow Q_H(\Gamma, G')$. The map $h_\#$ has closed image.

If h is injective, the map h_* is injective, identifying $\mathrm{Hom}(\Gamma, G)$ with its image. As $h_*(\mathrm{Hom}(\Gamma, G))$ is a closed invariant subscheme, by proposition 7, $t : h_*(\mathrm{Hom}(\Gamma, G)) \longrightarrow h_\#(Q_H(\Gamma, G))$ is a semi-geometric quotient, hence there exists a unique map $k : h_\#(Q_H(\Delta, G)) \longrightarrow Q_H(\Delta, G)$ such that $k \circ t = t \circ (h_\#)^{-1}$. This map is the inverse of $h_\#$, hence $h_\#$ is injective, and it identifies $Q_H(\Delta, G)$ with its image $h_\#(Q_H(\Delta, G)) \subset Q_H(\Gamma, G)$.

If h is bijective, then the map $h_\#$ is bijective. Even if h is surjective, the map $h_\#$ is not necessarily surjective. If Γ is a free group, and $h : G \longrightarrow H$ is surjective, then h_* is surjective. As $t \circ h_* = h_\# \circ t : \mathrm{Hom}(\Gamma, G) \longrightarrow Q_H(\Gamma, G')$, also $h_\#$ is surjective.

Let Δ be a finitely generated group, and let $k : \mathbb{Z}^{(m)} \longrightarrow \Delta$ be a presentation. Then $k^\# : Q_H(\Delta, G') \longrightarrow Q_H(\mathbb{Z}^{(m)}, G')$ identifies $Q_H(\Delta, G')$ with a closed subscheme $Y \subset Q_H(\mathbb{Z}^{(m)}, G')$, and the inverse image $X = (h_*)^{-1}(Y)$ through the map $h_* : Q_H(\mathbb{Z}^{(m)}, G) \longrightarrow \ \in Q_H(\mathbb{Z}^{(m)}, G')$ is a closed subscheme of $Q_H(\mathbb{Z}^{(m)}, G)$ with a canonical surjective map $X \longrightarrow Q_H(\Delta, H)$.

1.4.3 The case of representations

Everything becomes more explicit if we restrict to the more interesting case for us, when $G = GL_n, SL_n$ or $SL_n^{\pm}$, with the action by conjugation of $H = PGL_n$. The group G is affine, H is reductive, hence we are in the hypotheses above.

In this case we can describe the ring $A_0 \subset \mathcal{A}(\mathrm{Hom}(\Gamma, G))$ of invariant functions. Note that the trace functions τ_γ belong to A_0. Actually these functions generate A_0, and it is possible to find all the relations among them, hence we can have a complete description of the ring A_0 and of its spectrum $Q_{PGL_n}(\Gamma, G)$.

The case of a free group $\Gamma = \mathbb{Z}^{(m)}$, is studied in [Pr76]. There it is proven A_0 ring is generated, as a $\mathbb{Q}$-algebra, by the set $\{\tau_\gamma\}_{\gamma \in \Gamma}$. Hence there exists a finite set $C \subset \Gamma$ such that the functions $\{\tau_\gamma\}_{\gamma \in C}$ generate the ring A_0. Some of such C are described explicitly in [Pr76]. For example let $z_1, \ldots, z_m$ be a free set of generators of Γ, and let $C \subset \Gamma$ be the finite set of all non-commutative monomials in $z_1, \ldots, z_m$ of degree less than 2^n. The paper [Pr76] also finds all the relations between the trace functions, completing the description of $Q_{PGL_n}(\Gamma, G)$.

Let Γ be a finitely generated group and let $P : \mathbb{Z}^{(m)} \longrightarrow \Gamma$ be a presentation. The map $P_\# : Q_{PGL_n}(\Gamma, G) \longrightarrow Q_{PGL_n}(\mathbb{Z}^{(m)}, G)$, induced by P, is injective, identifying $Q_{PGL_n}(\Gamma, G)$ with a closed subscheme of $Q_{PGL_n}(\mathbb{Z}^{(m)}, G)$. Hence the ring $\mathcal{A}(Q_{PGL_n}(\Gamma, G))$ is a quotient of $\mathcal{A}(Q_{PGL_n}(\mathbb{Z}^{(m)}, G))$, hence if $C \subset \mathbb{Z}^{(m)}$ is a set such that the functions $\{\tau_\gamma\}_{\gamma \in C}$ generate the ring $\mathcal{A}(Q_{PGL_n}(\mathbb{Z}^{(m)}, G))$, then the functions $\{\tau_\gamma\}_{\gamma \in P(C)}$ is a set of functions generating $\mathcal{A}(Q_{PGL_n}(\Gamma, G))$.

Let Γ be a finitely generated group, and $C \subset \Gamma$ be a finite set such that the functions $\{\tau_\gamma\}_{\gamma \in C}$ generate the ring $A_0 = \mathcal{A}(Q_{PGL_n}(\Gamma, G))$.

Consider the map

$$t : \mathrm{Hom}(\Gamma, G(\mathbb{K})) \ni \rho \longrightarrow (\tau_\gamma(\rho))_{\gamma \in C} \in \mathbb{K}^{\mathrm{Card}\, C}$$

The image of this map is $Q_{PGL_n}(\Gamma, G(\mathbb{K}))$. The functions $\{\tau_\gamma\}_{\gamma \in C}$ determine the values of all the trace functions $\{\tau_\gamma\}_{\gamma \in \Gamma}$, hence, if ρ is a representation, the point $t(\rho)$ determines the character χ_ρ. Hence the points of $Q_{PGL_n}(\Gamma, G(\mathbb{K}))$ are in natural bijection with the characters of the representations in $\mathrm{Hom}(\Gamma, G(\mathbb{K}))$, and for this reason the algebraic set $Q_{PGL_n}(\Gamma, G(\mathbb{K}))$ will be called **varieties of characters**, and will be denoted by $\mathrm{Char}(\Gamma, G(\mathbb{K}))$, the set of $\mathbb{K}$-points of the algebraic scheme $\mathrm{Char}(\Gamma, G) = Q_{PGL_n}(\Gamma, G)$, the **scheme of characters**.

To avoid confusion, in the following we will denote by τ_γ the trace function relative to the element $\gamma \in \Gamma$ when considered as a function on the representation variety, and by I_γ the trace function relative to the element $\gamma \in \Gamma$ when considered as a function on the character variety.

Consider the invariant subsets $\mathrm{Hom}(\Gamma, G(\mathbb{K}))_{a.r.r.}$ and $\mathrm{Hom}(\Gamma, G(\mathbb{K}))_{a.i.r}$ of, respectively, absolutely reducible and absolutely irreducible representations. As $\mathrm{Hom}(\Gamma, G(\mathbb{K}))_{a.r.r.}$ is closed, its image through t is closed, and will be denoted by $\mathrm{Char}(\Gamma, G(\mathbb{K}))_{a.r.r.}$. As $\mathrm{Hom}(\Gamma, G(\mathbb{K}))_{a.i.r}$ is open, its images through t is open, and will be denoted by $\mathrm{Char}(\Gamma, G(\mathbb{K}))_{a.i.r.}$. By proposition 2, $\mathrm{Char}(\Gamma, G(\mathbb{K}))_{a.r.r.}$ and $\mathrm{Char}(\Gamma, G(\mathbb{K}))_{a.i.r.}$ are disjoint. They are the sets of $\mathbb{K}$-points of algebraic schemes $\mathrm{Char}(\Gamma, G)_{a.r.r.}$ and $\mathrm{Char}(\Gamma, G)_{a.i.r.}$.

Consider the restriction of t to $\mathrm{Hom}(\Gamma, G)_{a.i.r.}$:

$$t_{a.i.r.} : \mathrm{Hom}(\Gamma, G)_{a.i.r.} \longrightarrow \mathrm{Char}(\Gamma, G)_{a.i.r.}$$

This is a semi-geometric quotient, it is submersive, and, by proposition 2, the geometric fibers of t are precisely the orbits of the geometric points of $\mathrm{Hom}(\Gamma, G)_{a.i.r.}$, over an algebraically closed over-field $\mathbb{K} \supset k$. Hence this is a geometric quotient, $\mathrm{Hom}(\Gamma, G)_{a.i.r.} \subset \mathrm{Hom}(\Gamma, G)^s(\mathrm{Pre})$, and the set-theoretical quotient $\mathrm{Hom}(\Gamma, G(\mathbb{K}))_{a.i.r.}/PGL_n(\mathbb{K})$ is in natural bijection with $\mathrm{Char}(\Gamma, G(\mathbb{K}))_{a.i.r.}$.

Actually the action of PGL_n on the open subscheme $\mathrm{Hom}(\Gamma, G)_{a.i.r.}$ is free (see [Na00, corol. 6.5]) and $\mathrm{Hom}(\Gamma, G)_{a.i.r.} \subset \mathrm{Hom}(\Gamma, G)$ is precisely the subset of **properly stable points** for the action of PGL_n with respect to the canonical linearization of the trivial line bundle (see [MFK94, Chap. 1, def. 1.8] and [Na00, rem. 6.6]).

1.4.4 Real closed case

We need a similar construction for real closed fields. If $\mathbb{F} \supset k$ is a real closed over-field, and $G = GL_n$, SL_n or $SL_n^{\pm}$, the set of characters of representations $\rho : \Gamma \longrightarrow G(\mathbb{F})$ is not an affine algebraic set. Here we will show that this set is a closed semi-algebraic set, and that the map $t : \mathrm{Hom}(\Gamma, G(\mathbb{F})) \longrightarrow \mathrm{Char}(\Gamma, G(\mathbb{F}))$ has properties similar to the properties it has in the algebraically closed case.

Let $\mathbb{K} = \mathbb{F}[i]$, the algebraic closure of $\mathbb{F}$. As the representation varieties are defined over $\mathbb{Q}$, if $\mathrm{Hom}(\Gamma, G(\mathbb{K})) \subset \mathbb{K}^m$, we have $\mathrm{Hom}(\Gamma, G(\mathbb{F})) = \mathrm{Hom}(\Gamma, G(\mathbb{K})) \cap \mathbb{F}^n$, and if $\mathrm{Char}(\Gamma, G(\mathbb{K})) \subset \mathbb{K}^s$, we have $\mathrm{Char}(\Gamma, G(\mathbb{F})) = \mathrm{Char}(\Gamma, G(\mathbb{K})) \cap \mathbb{F}^s$.

The map $t : \mathrm{Hom}(\Gamma, G(\mathbb{K})) \longrightarrow \mathrm{Char}(\Gamma, G(\mathbb{K}))$ is defined over $\mathbb{Q}$, hence $t(\mathrm{Hom}(\Gamma, G(\mathbb{F})) \subset \mathrm{Char}(\Gamma, G(\mathbb{F}))$. Anyway $t(\mathrm{Hom}(\Gamma, G(\mathbb{F})))$ is not in general the whole $\mathrm{Char}(\Gamma, G(\mathbb{F}))$. For example an irreducible representation of Γ in $SU_2(\mathbb{C})$ has real character, but it is not conjugated to a representation in $SL_2(\mathbb{R})$ (see [MS84, prop. III.1.1] and the discussion for details). Hence the $\mathbb{F}$-algebraic set $\mathrm{Char}(\Gamma, G(\mathbb{F}))$ is not in a natural bijection with the set of characters of representations in $\mathrm{Hom}(\Gamma, G(\mathbb{F}))$. We will denote by $\overline{\mathrm{Char}}(\Gamma, G(\mathbb{F}))$ the image of $t_{|\,\mathrm{Hom}(\Gamma, G(\mathbb{F}))}$, the actual set of characters of representations in $\mathrm{Hom}(\Gamma, G(\mathbb{F}))$.

In the following we will consider $\mathbb{F}^n$ as a topological space with the topology inherited from the order of $\mathbb{F}$, hence the words closed and open refers to this topology. We will say Zariski closed and Zariski open if we want to refer to the Zariski topology.

Theorem 11. *Let $R \subset \mathrm{Hom}(\Gamma, G(\mathbb{F})) \subset \mathbb{F}^m$ be a closed semi-algebraic set that is invariant for the action of $PGL_n(\mathbb{F})$. Then the image $t(R)$ under the semi-geometric quotient map t is a closed semi-algebraic subset of $\mathbb{F}^s$.*

Proof : The idea of the proof is similar to the one of [CS83, prop. 1.4.4], but here we deal with semi-algebraic sets, instead of algebraic sets over an algebraically closed field.

The set $t(R)$ is semi-algebraic as it is the image of a semi-algebraic set via a polynomial map. To show that it is closed, let x_0 be a point in the closure of $t(R)$. Suppose, by contradiction, that x_0 is not in $t(R)$.

By the curve-selection lemma (see [BCR98, Chap. 2, thm. 2.5.5]) there exists a semi-algebraic map $f : [0, \varepsilon] \longrightarrow \mathbb{F}^{\mathrm{Card}(C)}$ such that $f(0) = x_0$ and $F = f((0, \varepsilon]) \subset \chi(R)$. The set F is a semi-algebraic curve.

Now we construct a semi-algebraic section $s : F \longrightarrow R$ such that for every $x \in F$, $t(s(x)) = x$. The set $D = t^{-1}(F)$ is a semi-algebraic subset of R. The map $t_{|D} : D \longrightarrow F$ induces a definable equivalence relation on D: $x \sim y \Leftrightarrow t(x) = t(y)$. By the existence of definable choice functions (see [Dr, Chap. 6, 1.2-1.3]) there exists a semi-algebraic map $h : D \longrightarrow D$ such that $h(x) = h(y) \Leftrightarrow t(x) = t(y)$, and a semi-algebraic map $k : h(D) \longrightarrow F$

such that $t = k \circ h$. The map k is bijective, and its inverse $k^{-1} = s$ is the searched section.

Consider the image $s(F) \subset R \subset \mathbb{F}^m$. If it is bounded in $\mathbb{F}^m$, then the closure S of $s(F)$ is contained in R, and $t(S)$ is closed (image of a closed, bounded set, see [Dr, Chap. 6, 1.10]) hence it contains x_0, a contradiction. Hence the image $s(F)$ is unbounded.

We embed $\mathbb{F}^m$ and $\mathbb{F}^s$ in the projective spaces $\mathbb{FP}^m$ and $\mathbb{RP}^s$. We denote by $\overline{R}$ the closure of R in $\mathbb{FP}^m$, by $\overline{t(R)}$ the closure of $t(R)$ in $\mathbb{RP}^s$ and by $\overline{t} : \overline{R} \longrightarrow \overline{t(R)}$ the extension of the map t, which exists because t is a polynomial map.

If we see $\mathbb{FP}^m$ as a closed and bounded algebraic subset of some $\mathbb{F}^M$, the image $s(F)$ is bounded in $\mathbb{F}^M$, its closure S for the order topology is contained in $\overline{R}$ and $\overline{t}(S)$ is, as before, closed, hence there is a point $y_0 \in S \subset \overline{R}$ such that $\overline{t}(y_0) = x_0$.

Let E be the Zariski closure of $s(F)$ in $\mathbb{FP}^m$. Up to restricting f to a smaller interval $[0, \varepsilon']$, we can suppose that E is an irreducible algebraic curve (see [BCR98, Chap. 2, prop. 2.8.2]) containing y_0. Let $E^{\mathbb{K}}$ be the extension of E to $\mathbb{KP}^m$. Let $\mathbb{F}(E)$ be the field of fractions of the curves E and $E^{\mathbb{K}}$, with coefficients in $\mathbb{F}$, and $\mathbb{K}(E^{\mathbb{K}})$ the field of fractions of $E^{\mathbb{K}}$ with coefficients in $\mathbb{K}$.

Let $\widetilde{E^{\mathbb{K}}}$ be the regular projective model of $E^{\mathbb{K}}$, i.e. a regular projective curve with a morphism $i : \widetilde{E^{\mathbb{K}}} \longrightarrow E^{\mathbb{K}}$ that is a birational isomorphism (see [Mu76, §7A, thm. 7.5]). We denote by $\widetilde{E}$ the inverse image $i^{-1}(E)$. Hence there is an isomorphism $\mathbb{K}(E^{\mathbb{K}}) \simeq \mathbb{K}(\widetilde{E^{\mathbb{K}}})$. Let $\widetilde{y_0}$ be an element of $i^{-1}(y_0)$. As $\widetilde{y_0}$ is a regular point, the local ring $\mathcal{O}_{\widetilde{y_0},\widetilde{E^{\mathbb{K}}}}$ is a UFD, and, as it has dimension 1, its maximal ideal $m_{\widetilde{y_0},\widetilde{E^{\mathbb{K}}}}$ is the unique prime ideal. Every irreducible element $\pi \in \mathcal{O}_{\widetilde{y_0},\widetilde{E^{\mathbb{K}}}}$ generate a prime ideal, hence $(\pi) = m_{\widetilde{y_0},\widetilde{E^{\mathbb{K}}}}$, hence all irreducible elements are associated. Let $v^{\mathbb{K}} : \mathbb{K}(E^{\mathbb{K}})^* \longrightarrow \mathbb{Z}$ be the π-adic valuation. We consider the restricted valuation $v : \mathbb{F}(E)^* \longrightarrow \mathbb{Z}$, and we denote $\mathcal{O} \subset \mathbb{F}(E)$ the valuation ring.

Let $\mathcal{R}_E : \Gamma \longrightarrow GL_n(\mathbb{F}(E))$, the canonical representation. We want to show that for every $\gamma \in \Gamma$, $\chi_{\mathcal{R}_E}(\gamma) \subset \mathcal{O}$, i.e. that all functions τ_γ don't have a pole in $\widetilde{y_0}$. It is enough to show this for the functions $\{\tau_\gamma\}_{\gamma \in C}$, as all other functions τ_γ are polynomials in these ones. As $t(y_0) = x_0$, a point in $\mathbb{F}^m$, we know that the functions $\{\tau_\gamma\}_{\gamma \in C}$ have a finite value in y_0, hence they don't have a pole in $\widetilde{y_0}$.

Hence $\chi_{\mathcal{R}_E} : \Gamma \longrightarrow \mathcal{O}$. This implies that this representation is conjugated, via $PGL_n(\mathbb{F}(E))$, with a representation $\mathcal{R}' : \Gamma \longrightarrow GL_n(\mathcal{O})$ (see the note after [MS84, prop. II.3.17]). Hence there exists a matrix $M = (m_{i,j}) \in GL_n(\mathbb{F}(E))$ such that for all $\gamma \in \Gamma$, $\mathcal{R}'(\gamma) = M\mathcal{R}_E(\gamma)M^{-1}$.

Let $U = E \cap \mathrm{Hom}(\Gamma, G(\mathbb{F})) \setminus \{$ poles of the functions $m_{i,j}\}$, a Zariski open subset of E. We define the map

$$d : U \ni \rho \longrightarrow M(\rho)\rho M(\rho)^{-1} \in \mathrm{Hom}(\Gamma, G(\mathbb{F}))$$

As for every $\rho \in U$, $d(\rho)$ is conjugated to ρ, then $t(\rho) = t(d(\rho))$. Moreover, for all $\rho \in U \cap R$, $d(\rho) \in R$, as R is invariant. The map d extends to a rational map $\widetilde{d} : \widetilde{E^{\mathbb{K}}} \longrightarrow \overline{\mathrm{Hom}(\Gamma, G(\mathbb{K}))}$, that is a morphism as $\widetilde{E^{\mathbb{K}}}$ is regular. Then $t \circ \widetilde{d}_{|\widetilde{E}} = t \circ i_{|\widetilde{E}}$, as this holds on an open subset. As the representation $\mathcal{R}'$ takes values in $GL_n(\mathcal{O})$, then $\widetilde{d}(y_0) \in \mathrm{Hom}(\Gamma, G(\mathbb{F}))$, and as $d(U \cap R) \subset R$, and y_0 is in the closure of U, then $\widetilde{d}(y_0) \in R$. We have $t(\widetilde{d}(y_0)) = t(i_{|\widetilde{E}}(y_0)) = x_0$, hence x_0 is in the image $t(R)$, a contradiction. □

Corollary 12. *The map*

$$t : \mathrm{Hom}(\Gamma, G(\mathbb{F})) \longrightarrow \overline{\mathrm{Char}}(\Gamma, G(\mathbb{F}))$$

has the following properties:

1. *The map t is surjective.*
2. *The set $\overline{\mathrm{Char}}(\Gamma, G(\mathbb{F}))$ is a closed semi-algebraic set in natural bijection with the set of characters of representations in $\mathrm{Hom}(\Gamma, G(\mathbb{F}))$.*
3. *If R is an invariant closed semi-algebraic subset of $\mathrm{Hom}(\Gamma, G(\mathbb{F}))$, then $t(R)$ is a closed semi-algebraic set.*
4. *If R is an invariant open semi-algebraic subset of $\mathrm{Hom}(\Gamma, G(\mathbb{F}))$, then $t(R)$ is a semi-algebraic set that is open in $\overline{\mathrm{Char}}(\Gamma, G(\mathbb{F}))$.*
5. *If V is an irreducible component of $\mathrm{Hom}(\Gamma, G(\mathbb{F}))$, then the image $t(V)$ is a closed semi-algebraic set.*
6. *The image $t(\mathrm{Hom}(\Gamma, G(\mathbb{F}))_{a.r.r.})$ is a closed semi-algebraic set, and it will be denoted by $\overline{\mathrm{Char}}(\Gamma, G(\mathbb{F}))_{a.r.r.}$.*
7. *The image $t(\mathrm{Hom}(\Gamma, G(\mathbb{F}))_{a.i.r.})$ is a semi-algebraic set that is open in $\overline{\mathrm{Char}}(\Gamma, G(\mathbb{F}))$ and it will be denoted by $\overline{\mathrm{Char}}(\Gamma, G(\mathbb{F}))_{a.i.r.}$.*
8. *The set $\overline{\mathrm{Char}}(\Gamma, G(\mathbb{F}))_{a.i.r.}$ is in natural bijection with the set theoretical quotient $\mathrm{Hom}(\Gamma, G(\mathbb{F}))_{a.i.r.}/PGL_n(\mathbb{F})$.*
9. *if $\{W_i\}_{i\in I}$ is a set of invariant closed subsets of $\mathrm{Hom}(\Gamma, G(\mathbb{F}))$, then*

$$t(\bigcap_{i\in I} W_i) = \bigcap_{i\in I} t(W_i)$$

10. *If $W_1, W_2 \subset \mathrm{Hom}(\Gamma, G(\mathbb{F}))$ are two closed disjoint invariant subsets, there exists an invariant function which is 0 on W_1 and 1 on W_2.*

□

Chapter 2

Spaces of convex projective structures

In this chapter we relate the material of the previous chapter with the theory of geometric structures on manifolds. Given a geometry $\mathbb{X} = (X, G)$, and a manifold M, we describe the deformation spaces of based $\mathbb{X}$-structures on M, denoted by $\mathcal{D}_{\mathbb{X}}(M)$, and marked $\mathbb{X}$-structures on M, denoted by $\mathcal{T}_{\mathbb{X}}(M) = \mathcal{D}_{\mathbb{X}}(M)/G$ (see below for all the definitions). We look for some geometry $\mathbb{X}$ and for some interesting subsets of $\mathcal{D}_{\mathbb{X}}(M)$ and $\mathcal{T}_{\mathbb{X}}(M)$ that can be naturally embedded as semi-algebraic subsets of, respectively, $\mathrm{Hom}(\pi_1(M), G)$ and $Q_G(\pi_1(M), G)$.

The first example, with $\mathbb{X} = \mathbb{H}^2$, the hyperbolic plane, is given by the sets of complete hyperbolic structures of finite area on a finite-type surface S. It follows from the results of Morgan and Shalen that the space of based hyperbolic structures of finite volume on a surface S is a closed semi-algebraic subset of $\mathrm{Hom}(\pi_1(M), SL_2(\mathbb{R}))$ and that space of corresponding marked hyperbolic structures is a closed semi-algebraic subset of $\overline{\mathrm{Char}}(\pi_1(M), SL_2(\mathbb{R}))$.

An other example, with $\mathbb{X} = \mathbb{RP}^n$, the projective space, is given by the set of convex projective structures on an n-manifold M. It has been recently proven by Benoist that, under suitable hypotheses on the fundamental group $\pi_1(M)$, the space of based convex projective structures on a n-manifold M is a union of connected components of $\mathrm{Hom}(\pi_1(M), SL_{n+1}(\mathbb{R}))$. Then it follows from corollary 12 that the space of corresponding marked projective structures is a union of connected components of $\overline{\mathrm{Char}}(\pi_1(M), SL_{n+1}(\mathbb{R}))$, hence a closed semi-algebraic subset.

2.1 Geometric structures

Let G be a $\mathbb{K}$-algebraic group, with $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. The Euclidean topology on $\mathbb{K}$ induces a topology on the set of $\mathbb{K}$-points $G(\mathbb{K})$, making it a Lie group. Most of what we are going to say holds for a general Lie group, but at some point, when dealing with representations in $G(\mathbb{K})$, the hypothesis that it is the set of $\mathbb{K}$-points of an algebraic group will be essential.

2.1.1 Definitions

A **geometry** is a pair $\mathbb{X} = (X, G(\mathbb{K}))$, where X is an analytic manifold, and $G(\mathbb{K})$ is endowed with a transitive analytic action $G(\mathbb{K}) \times X \longrightarrow X$.

If $U \subset X$ is open, a map $f : U \longrightarrow X$ is **locally-$\mathbb{X}$** if for each connected component $C \subset U$ there exists $g \in G(\mathbb{K})$ such that $g_{|C} = f_{|C}$. Note that f will be a local diffeomorphism. If $U \subset X$ is a connected non-empty open subset, and $f : U \longrightarrow X$ is locally-$\mathbb{X}$, then there exists a unique element $g \in G(\mathbb{K})$ such that $g_{|U} = f$. This property is called unique extension property.

Let M be a manifold. An **$\mathbb{X}$-atlas** on M is a pair $(\mathcal{U}, \Phi)$ where $\mathcal{U}$ is an open covering of M and $\Phi = \{\phi_U : U \longrightarrow X\}_{U \in \mathcal{U}}$ is a collection of **coordinate charts** such that for each pair $U, V \in \mathcal{U}$ the map

$$(\phi_U)_{|U \cap V} \circ (\phi_V^{-1})_{|\phi_V(U \cap V)} : \phi_V(U \cap V) \longrightarrow \phi_U(U \cap V) \subset X$$

is locally-$\mathbb{X}$.

An **$\mathbb{X}$-structure** on M is a maximal $\mathbb{X}$-atlas and an **$\mathbb{X}$-manifold** is a manifold together with an $\mathbb{X}$-structure on it. An $\mathbb{X}$-manifold inherits a real analytic structure from X. For example X itself is an $\mathbb{X}$ manifold, an atlas on X being made by the identity map alone. The same way, every open subset of X is an $\mathbb{X}$-manifold.

Let M, N be two $\mathbb{X}$-manifolds. A map $f : M \longrightarrow N$ is said to be an **$\mathbb{X}$-map** if for each pair of charts $\phi_U : U \longrightarrow X$, $\psi_V : V \longrightarrow X$ (where $U \subset M$ and $V \subset N$) the map

$$\psi_V \circ f_{|f^{-1}(V)} \circ (\phi_U^{-1})_{\phi(U \cap f^{-1}(V))} : \phi(U \cap f^{-1}(V)) \longrightarrow X$$

is locally-$\mathbb{X}$. Note that every $\mathbb{X}$-map is a local diffeomorphism. An invertible $\mathbb{X}$-map is a diffeomorphism, and its inverse is again an $\mathbb{X}$-map. Maps of this kind are called **$\mathbb{X}$-isomorphisms**. The set of self isomorphisms of an $\mathbb{X}$-manifold M is denoted by $\mathrm{Aut}_{\mathbb{X}}(M) = \mathrm{Aut}(M)$. For example $\mathrm{Aut}(X) = G(\mathbb{K})$. If $\Omega \subset X$ is a connected open subsets, then, by the unique extension property

$$\mathrm{Aut}_{\mathbb{X}}(\Omega) = \{g \in G(\mathbb{K}) \mid g(\Omega) = \Omega\}$$

If $f : M \longrightarrow N$ is a local diffeomorphism, then for every $\mathbb{X}$-structure on N there is a unique $\mathbb{X}$-structure on M making f an $\mathbb{X}$-map. In particular every covering space of an $\mathbb{X}$ manifold has a canonical $\mathbb{X}$-structure.

Vice versa if $f : M \longrightarrow N$ is a covering, M is an $\mathbb{X}$-manifold, and the deck transformations of M are $\mathbb{X}$-isomorphisms, then there is a unique $\mathbb{X}$-structure on N making f an $\mathbb{X}$-map. In other words, if M is an $\mathbb{X}$-manifold and $\Gamma \subset \mathrm{Aut}(M)$ is a discrete subgroup which acts properly and freely on

M, then $f : M \longrightarrow M/\Gamma$ is a covering and the quotient M/Γ has a canonical $\mathbb{X}$-structure.

2.1.2 Development maps

Let M be a simply connected $\mathbb{X}$-manifold. The local $\mathbb{X}$-structure on M can be globalized in the following sense. Every coordinate chart $\phi_U : U \longrightarrow X$ in the atlas can be extended in a unique way to an $\mathbb{X}$-map $D : M \longrightarrow X$. Such a map is called a **development map** for M. The development map is unique in the following sense: if $f : M \longrightarrow X$ is an $\mathbb{X}$-map, then there exists an $\mathbb{X}$-automorphism ψ of M and an element $g \in G(\mathbb{K})$ such that $f \circ \psi = g \circ D$. The development map completely determines the $\mathbb{X}$-structure on M.

Let M be an $\mathbb{X}$-manifold, and let $p : \widetilde{M} \longrightarrow M$ be its universal covering. The $\mathbb{X}$-structure on M induces an $\mathbb{X}$-structure on $\widetilde{M}$, with a developing map $D : \widetilde{M} \longrightarrow X$. We identify the fundamental group $\pi_1(M)$ with the group of deck transformations of covering space. Then there exists an homomorphism $h : \pi_1(M) \longrightarrow G(\mathbb{K})$ such that for every $\gamma \in \pi_1(M)$ we have $h(\gamma) \circ D = D \circ \gamma$. The pair (D, h) is called a **development pair**, and the homomorphism h is called **holonomy representation**. The development pair is unique in the following sense: if (D', h') is another such pair, there exist $g \in G(\mathbb{K})$ such that $D' = g \circ D$ and for all $\gamma \in \pi_1(M)$, $h'(\gamma) = gh(\gamma)g^{-1}$. A development pair determines the $\mathbb{X}$-structure on M.

If M is an $\mathbb{X}$-manifold and $m_0 \in M$, an $\mathbb{X}$-**germ** at m_0 is the germ of an $\mathbb{X}$-map from a neighborhood U of m_0 into X. The group $G(\mathbb{K})$ acts simply transitively on the set of $\mathbb{X}$-germs at m_0. Note that an $\mathbb{X}$-germ determines a unique developing map $D : \widetilde{M} \longrightarrow X$, and a unique holonomy representation $h : \pi_1(M, m_0) = \pi_1(M) \longrightarrow G(\mathbb{K})$.

Note that the development map $D : \widetilde{M} \longrightarrow X$ is a local homeomorphism. If it is injective, it identifies $\widetilde{M}$ with its image, an open set $\Omega \subset X$, and M is identified with a quotient $M = \Omega/\Gamma$, where $\Gamma = h(\pi_1(M))$, h being the corresponding holonomy map. In this case the holonomy representation is discrete and faithful and its image $\Gamma = h(\pi_1(M))$ acts properly and freely on Ω.

An $\mathbb{X}$-manifold M is said to be **complete** if the developing map $D : \widetilde{M} \longrightarrow X$ is a covering. Note that if X is simply connected, then D will be a diffeomorphism, in particular injective. As before, $\widetilde{M}$ is $\mathbb{X}$-isomorphic to X, and M is isomorphic to X/Γ, where $\Gamma = h(\pi_1(M))$.

2.1.3 Deformation spaces

Let S be a manifold. A **marked** $\mathbb{X}$-**structure** on S is a pair (M, ϕ), where M is an $\mathbb{X}$-manifold and $\phi : S \longrightarrow M$ is a diffeomorphism. The diffeomorphism ϕ induces an $\mathbb{X}$-structure on S. Two marked $\mathbb{X}$-structures (M, ϕ), (N, ψ) on

S are **isotopic** if there is an $\mathbb{X}$-map $h : M \longrightarrow N$ such that ψ is isotopic to $h \circ \phi$. Note that h is necessarily an isomorphism.

We choose a base point $s_0 \in S$ and a universal covering space $\widetilde{S} \longrightarrow S$. A **based** $\mathbb{X}$**-structure** on S is a triple (M, ϕ, ψ) where M is an $\mathbb{X}$-manifold, $\phi : S \longrightarrow M$ is a diffeomorphism and ψ is an $\mathbb{X}$-germ at $\phi(s_0)$. The diffeomorphism ϕ induces an $\mathbb{X}$-structure on S. The germ ψ determines a developing pair (D, h) for M, and this developing pair induces, via the diffeomorphism ϕ, a developing pair (f, ρ) for the $\mathbb{X}$-structure on S, such that $\rho : \pi_1(S, s_0) \longrightarrow G(\mathbb{K})$ is a representation, and $f : \widetilde{S} \longrightarrow X$ is a ρ-equivariant local diffeomorphism. Vice versa every such pair (f, ρ) determines a based $\mathbb{X}$-structure on S.

We say that two based $\mathbb{X}$-structures (f, ρ) and (f', ρ') are **isotopic** if $\rho = \rho'$ and there exists a diffeomorphism $h : (S, s_0) \longrightarrow (S, s_0)$, isotopic to the identity, such that $f' = f \circ \widetilde{h}$, where $\widetilde{h}$ is the lift of h to $\widetilde{S}$.

We consider the algebraic set $\mathrm{Hom}(\pi_1(S, s_0), G(\mathbb{K}))$ with the topology induced by the Euclidean topology of $\mathbb{K}$, and the set $C^\infty(\widetilde{S}, X)$ of smooth maps $\widetilde{S} \longrightarrow X$ with the C^∞ topology.

We define the deformation set of based $\mathbb{X}$-structures:

$$\mathcal{D}'_{\mathbb{X}}(S) = \{(f, \rho) \in C^\infty(\widetilde{S}, X) \times \mathrm{Hom}(\pi_1(S, s_0), G(\mathbb{K})) \mid$$

$$f \text{ is a } \rho\text{-equivariant local diffeomorphism}\}$$

This set inherits the subspace topology. We denote by $\mathrm{Diff}(S, s_0)$ the group of all diffeomorphisms $S \longrightarrow S$ fixing s_0, and by $\mathrm{Diff}_0(S, s_0)$ the subgroup of all diffeomorphisms fixing s_0 and isotopic to the identity. The group $\mathrm{Diff}_0(S, s_0)$ acts properly and freely on $\mathcal{D}'_{\mathbb{X}}(S)$. We denote by $\mathcal{D}_{\mathbb{X}}(S)$ the quotient by this action, the set of isotopy classes of based $\mathbb{X}$-structures:

$$\mathcal{D}_{\mathbb{X}}(S) = \mathcal{D}'_{\mathbb{X}}(S) / \mathrm{Diff}_0(S, s_0)$$

this set is endowed with the quotient topology. The group $G(\mathbb{K})$ acts on $\mathcal{D}'_{\mathbb{X}}(S)$ by composition on f and by conjugation on ρ, and this action passes to the quotient $\mathcal{D}_{\mathbb{X}}(S)$. We will denote the quotient by

$$\mathcal{T}_{\mathbb{X}}(S) = \mathcal{D}_{\mathbb{X}}(S) / G(\mathbb{K})$$

This set is endowed with the quotient topology. It is in natural bijection with the set of marked $\mathbb{X}$-structures up to isotopy.

The holonomy map

$$\mathrm{hol}' : \mathcal{D}'_{\mathbb{X}}(S) \ni (f, \rho) \longrightarrow \rho \in \mathrm{Hom}(\pi_1(S, s_0), G(\mathbb{K}))$$

is continuous and it is invariant under the action of $\mathrm{Diff}_0(S, s_0)$, hence it defines a continuous map

$$\mathrm{hol} : \mathcal{D}_{\mathbb{X}}(S) \longrightarrow \mathrm{Hom}(\pi_1(S, s_0), G(\mathbb{K}))$$

The group $G(\mathbb{K})$ acts on $\mathcal{D}_{\mathbb{X}}(S)$ as said, and it acts on $\mathrm{Hom}(\pi_1(S,s_0),G(\mathbb{K}))$ by conjugation. The map hol is equivariant with respect to these $G(\mathbb{K})$-actions, hence it induces a continuous map

$$\mathrm{hol} : \mathcal{T}_{\mathbb{X}}(S) \longrightarrow \mathrm{Hom}(\pi_1(S,s_0),G(\mathbb{K}))/G(\mathbb{K})$$

Theorem 13. [Deformation theorem] *The map* hol' *is open and the map*

$$\mathrm{hol} : \mathcal{D}_{\mathbb{X}}(S) \longrightarrow \mathrm{Hom}(\pi_1(S,s_0),G(\mathbb{K}))$$

is a local homeomorphism.

Proof : See [Th, 5.3.1], [Lo84] or [CEG87]. □

To prove that also $\mathrm{hol} : \mathcal{T}_{\mathbb{X}}(S) \longrightarrow \mathrm{Hom}(\pi_1(S,s_0),G(\mathbb{K}))/G(\mathbb{K})$ is a local homeomorphism we need additional hypotheses.

If G is reductive, the set of **stable points** $\mathrm{Hom}(\pi_1(S,s_0),G)^s \subset \mathrm{Hom}(\pi_1(S,s_0),G)$ (see [MFK94, Chap. 1, def. 1.7]) is a Zariski open invariant subset upon which the action of G is proper. For example, if $G = GL_n$, the set $\mathrm{Hom}(\pi_1(S,s_0),GL_n)_{a.i.r.} \subset \mathrm{Hom}(\pi_1(S,s_0),GL_n)^s$.

Theorem 14. *We denote* $\mathcal{T}_{\mathbb{X}}(S)^s = \mathrm{hol}^{-1}(\mathrm{Hom}(\pi_1(S,s_0),G(\mathbb{K}))^s)$*. Then the map:*

$$\mathrm{hol} : \mathcal{T}_{\mathbb{X}}(S)^s \longrightarrow \mathrm{Hom}(\pi_1(S,s_0),G(\mathbb{K}))^s/G(\mathbb{K})$$

is a local homeomorphism

Proof : See [G, cor. 3.2]. □

2.2 Hyperbolic structures

2.2.1 Definition

Let B be a symmetric bilinear form on $\mathbb{R}^{n+1}$ with signature $[1,n]$:

$$B = \begin{pmatrix} 1 & & & & \\ & -1 & & & \\ & & -1 & & \\ & & & \ddots & \\ & & & & -1 \end{pmatrix}$$

The **Lorenz group** is the orthogonal group of B:

$$O(1,n) = \{A \in GL_{n+1}(\mathbb{R}) \mid A^t B A = \mathrm{Id}\}$$

an algebraic subgroup of $GL_{n+1}(\mathbb{R})$. The elements of $O(1,n)$ have determinant 1 or -1. We will denote by $SO(1,n)$ the algebraic subgroup of $O(1,n)$ consisting of matrices with determinant 1.

The set of **time-like vectors** is

$$\widetilde{\Omega} = \{v \in \mathbb{R}^{n+1} \mid v^t B v > 0\}$$

This is an open cone in $\mathbb{R}^{n+1}$, whose boundary is the set of **isotropic vectors**:

$$\widetilde{\partial\Omega} = \{v \in \mathbb{R}^{n+1} \mid v^t B v = 0\}$$

a closed cone.

The open cone $\widetilde{\Omega}$ is the union of two open convex cones

$$\widetilde{\Omega}^+ = \{v \in \widetilde{\Omega} \mid (1, 0, \ldots, 0) B v > 0\}$$

$$\widetilde{\Omega}^- = \{v \in \widetilde{\Omega} \mid (1, 0, \ldots, 0) B v < 0\}$$

$\widetilde{\Omega}$ is invariant for the action of the Lorenz group $O(1, n)$. Let

$$O^+(1, n) = \{A \in O(1, n) \mid A(\widetilde{\Omega}^+) \subset \widetilde{\Omega}^+\}$$

This is the identity component of $O(1, n)$, a subgroup of index 2. Similarly, let

$$SO^+(1, n) = \{A \in SO(1, n) \mid A(\widetilde{\Omega}^+) \subset \widetilde{\Omega}^+\}$$

the identity component of $SO(1, n)$, a subgroup of index 2.

Let $\pi : \mathbb{R}^{n+1} \longrightarrow \mathbb{RP}^n$ be the natural projection. Set $\Omega = \pi(\widetilde{\Omega}) \subset \mathbb{RP}^n$ and $\partial\Omega = \pi(\widetilde{\partial\Omega}) \subset \mathbb{RP}^n$. Note that Ω and $\partial\Omega$ are invariant for the natural action of the group $O(1, n)$ on $\mathbb{RP}^n$.

Let $\mathbb{H}^n = (\Omega, O^+(1, n))$, the **hyperbolic space** (or the **Klein model** of the hyperbolic space).

An **hyperbolic structure** is an $\mathbb{H}^n$-structure, and an **hyperbolic manifold** is an $\mathbb{H}^n$-manifold.

Note that the group $O^+(1, n)$ acts properly on $\mathbb{H}^n$, hence every discrete subgroup of $O^+(1, n)$ acts properly on $\mathbb{H}^n$.

2.2.2 Deformation spaces

Let $\overline{S} = \Sigma_g^k$ be an orientable compact surface of genus g, with $k \geq 0$ boundary components and with $\chi(S) < 0$, and let S be its interior. As S is orientable, its fundamental group $\pi_1(S)$ is torsion-free, moreover the image of the holonomy of an hyperbolic structure on S only contains orientation preserving maps, hence it takes values in $SO^+(1, 2)$. Note that the group $O^+(1, 2)$ is isomorphic, as a $\mathbb{R}$-algebraic group, to the group $PGL_2(\mathbb{R})$, and $SO^+(1, 2)$ is isomorphic, as a $\mathbb{R}$-algebraic group, to the group $PSL_2(\mathbb{R})$, hence we can think the holonomy of an hyperbolic structure on S as a representation in $PSL_2(\mathbb{R})$.

Suppose first that S is a closed surface ($k = 0$), and consider the spaces $\mathcal{D}_{\mathbb{H}^2}(S)$ and $\mathcal{T}_{\mathbb{H}^2}(S)$. A theorem assures that every closed hyperbolic manifold is automatically complete. As a complete structure is determined by its holonomy, the map

$$\mathrm{hol} : \mathcal{D}_{\mathbb{H}^2}(S) \longrightarrow \mathrm{Hom}(\pi_1(S), PSL_2(\mathbb{R}))$$

is injective, and, by the deformation theorem, it identifies the set $\mathcal{D}_{\mathbb{H}^2}(S)$ with an open subset of $\mathrm{Hom}(\pi_1(M), PSL_2(\mathbb{R}))$, and $\mathcal{T}_{\mathbb{H}^2}(S)$ is identified with an open subset of $\mathrm{Hom}(\pi_1(M), PSL_2(\mathbb{R}))/PGL_2(\mathbb{R})$. The representations in $\mathcal{D}_{\mathbb{H}^2}(S)$ are precisely the discrete and faithful representations, because, as $\pi_1(S)$ is torsion-free, all discrete and faithful representations acts freely on $\mathbb{H}^2$.

If S is not closed ($k > 0$), it is convenient to add additional requirements on the studied structures. Consider the subsets $\mathcal{D}^{cf}_{\mathbb{H}^2}(S) \subset \mathcal{D}_{\mathbb{H}^2}(S)$ and $\mathcal{T}^{cf}_{\mathbb{H}^2}(S) \subset \mathcal{T}_{\mathbb{H}^2}(S)$ of (respectively) based and marked hyperbolic structures on S that are complete and with finite volume. These spaces have properties that are similar to the closed case, the map

$$\mathrm{hol} : \mathcal{D}^{cf}_{\mathbb{H}^2}(S) \longrightarrow \mathrm{Hom}(\pi_1(S), PSL_2(\mathbb{R}))$$

is injective and it identifies $\mathcal{D}^{cf}_{\mathbb{H}^2}(S)$ with a subset of $\mathrm{Hom}(\pi_1(S), PSL_2(\mathbb{R}))$, and $\mathcal{T}^{cf}_{\mathbb{H}^2}(S)$ with a subset of $\mathrm{Hom}(\pi_1(S), PSL_2(\mathbb{R}))/PGL_2(\mathbb{R})$.

We choose elements $\beta_1, \ldots, \beta_k \in \pi_1(S)$ corresponding to the boundary components of $\overline{S}$. An hyperbolic structure on S is complete and with finite volume if and only if its holonomy h is a discrete and faithful representation, and the absolute values of the traces of the matrices $h(\beta_i) \in PSL_2(\mathbb{R})$ is 2. Conversely all such representations correspond to complete hyperbolic structures with finite volume.

To have a uniform notation, if $k = 0$ we will write $\mathcal{D}^{cf}_{\mathbb{H}^2}(S) = \mathcal{D}_{\mathbb{H}^2}(S)$ and $\mathcal{T}^{cf}_{\mathbb{H}^2}(S) = \mathcal{T}_{\mathbb{H}^2}(S)$. The spaces $\mathcal{T}^{cf}_{\mathbb{H}^2}(S)$ are usually called **Teichmüller spaces**. By the arguments above we have:

Proposition 15. *The deformation space $\mathcal{D}^{cf}_{\mathbb{H}^2}(S)$ is identified in a natural way with a subset of* $\mathrm{Hom}(\pi_1(S), PSL_2(\mathbb{R}))$, *and the Teichmüller space $\mathcal{T}^{cf}_{\mathbb{H}^2}(S)$ is identified in a natural way with a subset of the quotient* $\mathrm{Hom}(\pi_1(S), PSL_2(\mathbb{R}))/PGL_2(\mathbb{R})$. □

2.2.3 Character spaces

Now we want to show that the Teichmüller spaces can be embedded in $\mathrm{Char}(\pi_1(S), SL_2(\mathbb{R}))$, with an identification that is natural only up to the action of the group $\mathrm{Hom}(\pi_1(S), \mathbb{Z}_2)$. As in the previous subsection, let $\overline{S} = \Sigma_g^k$ be an orientable compact surface of genus g, with $k \geq 0$ boundary components and with $\chi(S) < 0$, and let S be its interior. Let $\pi : SL_2(\mathbb{R}) \longrightarrow PSL_2(\mathbb{R})$ denote the quotient.

Proposition 16. *The image of the map*

$$\pi_* : \mathrm{Hom}(\pi_1(S), SL_2(\mathbb{R})) \longrightarrow \mathrm{Hom}(\pi_1(S), PSL_2(\mathbb{R}))$$

contains the deformation space $\mathcal{D}^{cf}_{\mathbb{H}^2}(S) \subset \mathrm{Hom}(\pi_1(S), PSL_2(\mathbb{R}))$.

Proof : If $k > 0$, the map π_* is surjective, as in this case $\pi_1(S)$ is free. If $k = 0$, let $\rho : \pi_1(S) \longrightarrow PSL_2(\mathbb{R})$ be a discrete and faithful representation. By theorem [CS83, Prop. 3.1.1], the inverse image $\pi^{-1}(\rho(\pi_1(S)))$ is isomorphic to $\pi_1(S) \times \mathbb{Z}_2$, hence there exists a lift $\overline{\rho} : \pi_1(S) \longrightarrow SL_2(\mathbb{R})$ such that $\rho = \pi \circ \overline{\rho} = \pi_*(\overline{\rho})$. □

Let $\rho : \pi_1(S) \longrightarrow PSL_2(\mathbb{R})$ be a representation, and suppose that there exists an element $\overline{\rho} \in {\pi_*}^{-1}(\rho)$. Then for every element $s \in \mathrm{Hom}(\pi_1(S), \mathbb{Z}_2)$ there is another lift:

$$\overline{\rho}^s : \pi_1(S) \ni \gamma \longrightarrow (-1)^s \overline{\rho}(\gamma) \in SL_2(\mathbb{R})$$

hence the inverse image $\pi_*^{-1}(\rho)$ of a representation ρ is empty or it has the same cardinality as $\mathrm{Hom}(\pi_1(S), \mathbb{Z}_2)$, i.e. the rank of the abelian group $\pi_1(S)/[\pi_1(S), \pi_1(S)]$.

If $k > 0$, we choose elements $\beta_1, \dots, \beta_k \in \pi_1(S)$ corresponding to the boundary components of $\overline{S}$. For every element $\gamma \in \pi_1(S)$ the function τ_γ is a polynomial function on the algebraic set $\mathrm{Hom}(\pi_1(S), SL_2(\mathbb{R}))$. We define the algebraic subset

$$\mathrm{Hom}^p(\pi_1(S), SL_2(\mathbb{R})) = \{\rho \mid \forall i \in \{1 \dots k\} : \tau_{\beta_i}(\rho) = \pm 2\}$$

If $k = 0$ we will write $\mathrm{Hom}^p(\pi_1(S), SL_2(\mathbb{R})) = \mathrm{Hom}(\pi_1(S), SL_2(\mathbb{R}))$.

We denote by $\overline{\mathcal{D}^{cf}_{\mathbb{H}^2}(S)}$ the subset of all discrete and faithful representations in $\mathrm{Hom}^p(\pi_1(S), SL_2(\mathbb{R}))$.

Proposition 17. *The map* $\pi_* : \overline{\mathcal{D}^{cf}_{\mathbb{H}^2}(S)} \longrightarrow \mathcal{D}^{cf}_{\mathbb{H}^2}(S)$ *is surjective, and it is the quotient of* $\overline{\mathcal{D}^{cf}_{\mathbb{H}^2}(S)}$ *by the action of the group* $\mathrm{Hom}(\pi_1(S), \mathbb{Z}_2)$*. The set* $\overline{\mathcal{D}^{cf}_{\mathbb{H}^2}(S)}$ *is a finite union of connected components of* $\mathrm{Hom}^p(\pi_1(S), SL_2(\mathbb{R}))$*, and the action of* $\mathrm{Hom}(\pi_1(S), \mathbb{Z}_2)$ *on the set of connected components of* $\overline{\mathcal{D}^{cf}_{\mathbb{H}^2}(S)}$ *is free and transitive, identifying* $\mathcal{D}^{cf}_{\mathbb{H}^2}(S)$ *with one of them. In particular* $\mathcal{D}^{cf}_{\mathbb{H}^2}(S)$ *is identified with a clopen semi-algebraic subset of* $\mathrm{Hom}^p(\pi_1(S), SL_2(\mathbb{R}))$*. Moreover the signs of the trace functions* τ_γ *are constant on the connected components of* $\overline{\mathcal{D}^{cf}_{\mathbb{H}^2}(S)}$*.*

Proof : By [MS84, Prop. III.1.6], the set $\overline{\mathcal{D}^{cf}_{\mathbb{H}^2}(S)}$ is open and closed in $\mathrm{Hom}^p(\pi_1(S), SL_2(\mathbb{R}))$.

Note that matrices in $SL_2(\mathbb{R})$ with null trace have order 4, but $\pi_1(S)$ is torsion-free, hence if $\rho \in \overline{\mathcal{D}^{cf}_{\mathbb{H}^2}(S)}$, then for all $\gamma \in \pi_1(S)$, $\mathrm{tr}(\rho(\gamma)) \neq 0$. Hence the signs of the trace functions τ_γ are constant on the connected components of $\overline{\mathcal{D}^{cf}_{\mathbb{H}^2}(S)}$, hence the group $\mathrm{Hom}(\pi_1(S), \mathbb{Z}_2)$ acts freely on the finite set of connected components of $\overline{\mathcal{D}^{cf}_{\mathbb{H}^2}(S)}$.

As, by Teichmüller theory, the space $\mathcal{D}^{cf}_{\mathbb{H}^2}(S)$ is connected, the action of $\mathrm{Hom}(\pi_1(S), \mathbb{Z}_2)$ on the set of connected components of $\overline{\mathcal{D}^{cf}_{\mathbb{H}^2}(S)}$ is transitive as well. □

We will always identify $\mathcal{D}^{cf}_{\mathbb{H}^2}(S)$ with a subset of $\mathrm{Hom}^p(\pi_1(S), SL_2(\mathbb{R}))$, even if this identification is not canonical.

In a similar way, $\mathcal{T}^{cf}_{\mathbb{H}^2}(S)$ can be identified with a semi-algebraic subset of $\overline{\mathrm{Char}}(\pi_1(S), SL_2(\mathbb{R}))$. We define the algebraic subset

$$\mathrm{Char}^p(\pi_1(S), SL_2(\mathbb{R})) = \{\chi \mid \forall i \in \{1 \dots k\} : I_{\beta_i}(\chi) = \pm 2\}$$

and the closed semi-algebraic subset

$$\overline{\mathrm{Char}}^p(\pi_1(S), SL_2(\mathbb{R})) = \mathrm{Char}^p(\pi_1(S), SL_2(\mathbb{R})) \cap \overline{\mathrm{Char}}(\pi_1(S), SL_2(\mathbb{R}))$$

Note that $t(\mathrm{Hom}^p(\pi_1(S), SL_2(\mathbb{R}))) = \overline{\mathrm{Char}}^p(\pi_1(S), SL_2(\mathbb{R}))$, where t is the semi-geometric quotient map, and $\mathrm{Hom}^p(\pi_1(S), SL_2(\mathbb{R}))$ is closed and invariant for the action by conjugation of $PGL_2(\mathbb{R})$.

Proposition 18. *The image $t(\mathcal{D}^{cf}_{\mathbb{H}^2}(S))$ can be identified with the Teichmüller space $\mathcal{T}^{cf}_{\mathbb{H}^2}(S)$, it is a connected component and, in particular, a clopen semi-algebraic subset of both* $\mathrm{Char}^p(\pi_1(S), SL_2(\mathbb{R}))$ *and* $\overline{\mathrm{Char}}^p(\pi_1(S), SL_2(\mathbb{R}))$.

Proof : As $\chi(S) < 0$, the group $\pi_1(S)$ is non-solvable, hence the discrete and faithful representations of $\pi_1(S)$ in $SL_2(\mathbb{R})$ are absolutely irreducible. Hence $\mathcal{D}^{cf}_{\mathbb{H}^2}(S) \subset \mathrm{Hom}(\pi_1(S), SL_2(\mathbb{R}))_{a.i.r}$, the invariant open subset of absolutely irreducible representations. Hence the image $t(\mathcal{D}^{cf}_{\mathbb{H}^2}(S))$ can be identified with the quotient $\mathcal{D}^{cf}_{\mathbb{H}^2}(S)/PGL_2(\mathbb{R}) = \mathcal{T}^{cf}_{\mathbb{H}^2}(S)$.

The subset $\mathcal{D}^{cf}_{\mathbb{H}^2}(S) \subset \mathrm{Hom}^p(\pi_1(S), SL_2(\mathbb{R}))$ is invariant, semi-algebraic and clopen, hence, by corollary 12 its image $\mathcal{T}^{cf}_{\mathbb{H}^2}(S)$ is also semi-algebraic and clopen in $\overline{\mathrm{Char}}^p(\pi_1(S), SL_2(\mathbb{R}))$. It is also open in $\mathrm{Char}^p(\pi_1(S), SL_2(\mathbb{R}))$, see [MS84, prop. III.1.8]. □

2.2.4 Trace functions

Consider the family of functions $\mathcal{G} = \{I_\gamma\}_{\gamma \in \pi_1(S)}$. These are functions on $\mathrm{Char}(\pi_1(S), SL_2(\mathbb{R}))$. The immersion of the Teichmüller space $\mathcal{T}^{cf}_{\mathbb{H}^n}(S)$ in $\mathrm{Char}(\pi_1(S), SL_2(\mathbb{R}))$ is not canonical, it is well defined only up to the action of the group $\mathrm{Hom}(\pi_1(S), \mathbb{Z}_2)$. If two points of $\mathrm{Char}(\pi_1(S), SL_2(\mathbb{R}))$ have the same orbit for the action of $\mathrm{Hom}(\pi_1(S), \mathbb{Z}_2)$, the values of functions I_γ on these points coincide up to sign. Hence only $|I_c|$ is a well defined function on $\mathcal{T}^{cf}_{\mathbb{H}^n}(S)$.

As the functions I_γ never vanish on $\mathcal{T}^{cf}_{\mathbb{H}^2}(S)$, and $\mathcal{T}^{cf}_{\mathbb{H}^2}(S)$ is connected, they have constant sign. We define the **positive trace functions** by choosing a point $x \in \mathcal{T}^{cf}_{\mathbb{H}^2}(S)$ and then defining for every $\gamma \in \pi_1(S)$ the function $J_\gamma = \operatorname{sign}(I_\gamma(x)) I_\gamma$.

On $\mathcal{T}^{cf}_{\mathbb{H}^2}(S)$ we have $J_\gamma = |I_\gamma|$, hence the functions J_γ are canonically well defined on $\mathcal{T}^{cf}_{\mathbb{H}^2}(S)$, and they are polynomial functions on $\operatorname{Char}(\pi_1(S), SL_2(\mathbb{R}))$, generating the ring of coordinates.

Positive trace functions are closely related to length functions:

$$\ell_\gamma([h]) = \inf_\alpha l_h(\alpha)$$

where h is an hyperbolic metric on S, $[h]$ is the corresponding elements of $\mathcal{T}^{cf}_{\mathbb{H}^2}(S)$, l_h is the function sending a curve in its h-length, and the inf is taken on the set of all closed curves whose free homotopy class is γ. The relation between trace functions and length function is given by

$$J_\gamma([h]) = 2\cosh(\frac{1}{2}\ell_c([h]))$$

In particular this implies that $|I_c([h])| \geq 2$ on $\mathcal{T}^{cf}_{\mathbb{H}^2}(S)$.

2.3 Projective structures

2.3.1 Convex Projective Structures

Let $\mathbb{RP}^n = (\mathbb{RP}^n, PGL_{n+1}(\mathbb{R}))$. A **projective structure** is an $\mathbb{RP}^n$-structure, and a **projective manifold** is a $\mathbb{RP}^n$-manifold.

An **affine space** in $\mathbb{RP}^n$ is the complement of a projective hyperplane. A set $\Omega \subset \mathbb{RP}^n$ is **convex** if it is contained in some affine space and its intersection with every projective line is connected. A convex set is **properly convex** if its closure $\overline{\Omega}$ is contained in an affine space.

A **convex projective manifold** is a projective manifold isomorphic to Ω/Γ, where $\Omega \subset \mathbb{RP}^n$ is an open properly convex domain and $\Gamma \subset PGL_{n+1}(\mathbb{R})$ is a discrete group acting properly and freely on Ω. In other words, a projective structure is convex if an only if the developing map is injective, with image a properly convex open subset of $\mathbb{RP}^n$.

For example, $\mathbb{H}^n \subset \mathbb{RP}^n$ is an open, strictly convex set, $O^+(1,n) \subset PGL_{n+1}(\mathbb{R})$, hence every complete hyperbolic manifold is a convex projective manifold.

Let $\Omega \subset \mathbb{RP}^n$ be an open properly convex set. The group of **projective automorphisms** of Ω is the group of $\mathbb{RP}^n$-automorphisms:

$$\operatorname{Aut}(\Omega) = \operatorname{Aut}_{\mathbb{RP}^n}(\Omega) = \{g \in PGL_{n+1}(\mathbb{R}) \mid g(\Omega) = \Omega\}$$

The group $\mathrm{Aut}(\Omega)$ acts properly on Ω (see [Be0, sez. 2.1]), hence every discrete subgroup of $\mathrm{Aut}(\Omega)$ acts properly on Ω.

A subgroup $\Gamma \subset \mathrm{Aut}(\Omega)$ **divides** Ω if it is discrete and the quotient Ω/Γ is compact. If such a group exists, Γ is called a **divisible convex set**. If Γ is torsion-free, the action is free, hence the quotient Ω/Γ is a closed manifold, with a convex projective structure.

If Γ divides Ω, then it is finitely generated, hence, by Selberg lemma, it has a torsion-free subgroup Γ' of finite index. The subgroup Γ' again divides Ω, and Ω/Γ' is a closed manifold covering the closed orbifold Ω/Γ.

Let Γ be a group. The **virtual center** of Γ is the subgroup of all elements of Γ whose centralizer has finite index in Γ. The virtual center of Γ is trivial if and only if every subgroup with finite index in Γ has trivial center.

A subgroup $\Gamma \subset PGL_{n+1}(\mathbb{R})$ is **strongly irreducible** if and only if all subgroups of finite index of Γ are irreducible.

Proposition 19. *If $\Gamma \subset PGL_{n+1}(\mathbb{R})$ divides an open properly convex set Ω, then Γ has trivial virtual center if and only if it is strongly irreducible.*

Proof : See [Be0] and [Be3]. □

A properly convex set $\Omega \subset \mathbb{RP}^n$ is said to be **reducible** if there exists a direct-sum decomposition $\mathbb{R}^{n+1} = V \oplus W$ such that the cone $\overline{\Omega} = \pi^{-1}(\Omega) \cup \{0\} \subset \mathbb{R}^{n+1}$ is the direct sum $C + D$ of two convex cones $C \subset V$, $D \subset W$, otherwise it is said to be **irreducible**.

Proposition 20. [Vey's irreducibility theorem] *If $\Gamma \subset PGL_{n+1}(\mathbb{R})$ divides an open properly convex set Ω, then Γ is strongly irreducible if and only if Ω is irreducible.*

Proof : See [Ve70]. □

As a corollary, if M is a complete hyperbolic manifold, the fundamental group of M has trivial virtual center, as it is isomorphic to a subgroup of $O^+(1,n)$ dividing $\mathbb{H}^n$, that is irreducible.

2.3.2 Strictly convex projective structures

A properly convex set $\Omega \subset \mathbb{RP}^n$ is **strictly convex** if its boundary $\partial\Omega$ does not contain any segment. A **strictly convex projective manifold** is a convex projective manifold Ω/Γ, where Ω is strictly convex. The following results about strictly convex divisible sets have been proved in [Be1].

Proposition 21. *Let $\Gamma \subset PGL_{n+1}(\mathbb{R})$ be a group dividing an open properly convex set Ω. Then Ω is strictly convex if and only if the group Γ is Gromov hyperbolic.*

Proof : See [Be1]. □

As this property depends only on the abstract group Γ, if the fundamental group of a manifold M is Gromov hyperbolic, then all convex projective structures on M are strictly convex.

Proposition 22. *A properly convex divisible set has a boundary of class C^1 if and only if it is strictly convex.*

Proof : See [Be1] $\square$

For example every closed hyperbolic n-manifold is a strictly convex projective manifold.

Proposition 23. *Let $n : \partial\Omega \longrightarrow \mathbb{S}^{n-1}$ be the normal map. Then if Ω is a strictly convex divisible set, then there exists an α, such that $0 < \alpha < 1$ and $\partial\Omega$ has regularity $C^{1+\alpha}$ (i.e. the map n is α-Hölder). If n is absolutely continuous, or if n is α-Hölder for all $\alpha < 1$, then $\Omega \simeq \mathbb{H}^n$.*

Proof : See [Be1] $\square$

Let $SL^{\pm}_{n+1}(\mathbb{R}) \subset GL_{n+1}(\mathbb{R})$ be the subgroup of matrices with determinant ± 1. Then $PGL_{n+1} = SL^{\pm}_{n+1}(\mathbb{R})/\{\pm \mathrm{Id}\}$.

If $\gamma \in PGL_{n+1}(\mathbb{R})$, let $\overline{\gamma} \in SL^{\pm}_{n+1}(\mathbb{R})$ be a lift. Let $\lambda_1(\gamma), \ldots, \lambda_{n+1}(\gamma)$ be its complex eigenvalues, ordered such that $|\lambda_1(\gamma)| \geq |\lambda_2(\gamma)| \geq \cdots \geq |\lambda_{n+1}(\gamma)|$. The element γ is said to be **proximal** if $|\lambda_1(\gamma)| > |\lambda_2(\gamma)|$. In this case $\lambda_1(\gamma)$ is real, and its eigenvector corresponds to the unique attracting fixed point $x_g \in \mathbb{RP}^n$ of g.

Proposition 24. *Let $\Gamma \subset PGL_{n+1}(\mathbb{R})$ be a torsion-free group dividing a strictly convex set Ω. Then every element $\gamma \in \Gamma$ is proximal. In particular γ^{-1} is also proximal, hence the eigenvector $\lambda_{n+1}(\gamma)$ is real. Moreover, if $\overline{\gamma} \in SL^{\pm}_{n+1}(\mathbb{R})$ is a lift of γ, then $\lambda_1(\overline{\gamma})$ and $\lambda_{n+1}(\overline{\gamma})$ have the same sign.*

Proof : See [Be1]. $\square$

The point $y_\gamma = x_{\gamma^{-1}}$ is the unique repelling fixed point of γ. The points x_γ, y_γ are in $\partial\Omega$, and the segment $[x_g, y_g]$ is the unique invariant geodesic of γ in Ω. The image of $[x_\gamma, y_\gamma]$ in Ω/Γ is the unique geodesic in the free-homotopy class of γ. Moreover, Ω/Γ does not contain any closed homotopically trivial geodesic.

Corollary 25. *The set $\pi^{-1}(\Omega) \subset \mathbb{R}^{n+1}$ is the union of two convex cones. The group Γ can be lifted to a subgroup $\overline{\Gamma}$ of $SL^{\pm}_{n+1}(\mathbb{R})$ preserving each of the convex cones. After this lift, if $\gamma \in \Gamma$, then λ_1 and λ_{n+1} are real and positive.* $\square$

Proposition 26. *Let $\Gamma \subset PGL_{n+1}(\mathbb{R})$ be a torsion-free group dividing a strictly convex set Ω. The set $\{x_\gamma \mid \gamma \in \Gamma\}$ is dense in $\partial\Omega$, hence Ω is actually determined by Γ.*

Proof : See [Be1] $\square$

Proposition 27. *Let $\Gamma \subset PGL_{n+1}(\mathbb{R})$ be a group dividing a strictly convex set Ω. Then Γ is absolutely irreducible, i.e. it has no nontrivial invariant subspaces in $\mathbb{CP}^n$.*

Proof : If $\Omega \not\simeq \mathbb{H}^n$, by [Be2], the Zariski closure of Γ is $PGL_{n+1}(\mathbb{R})$. If $\Omega \simeq \mathbb{H}^n$, the Zariski closure of Γ is $O^+(1,n)$. The linear span of $\overline{\Gamma} \subset M_n(\mathbb{R})$ contains the Zariski closure, hence it is $M_n(\mathbb{R})$, hence the group Γ is absolutely irreducible. □

2.3.3 Hilbert metric

Let $\Omega \subset \mathbb{RP}^n$ be a properly convex set. Hilbert defined a metric on Ω that is invariant by projective automorphisms of Ω. This metric is defined by using cross-ratios: if $x, y \in \Omega$, the projective line through x and y intersects $\partial\Omega$ in two points a, b. The distance is then defined as

$$d_\Omega(x,y) = \frac{1}{2}\log[a,x,y,b]$$

where the order is chosen such that $\overline{ax} \cap \overline{yb} = \emptyset$. If we add the condition $d_\Omega(x,x) = 0$, the function $d : \Omega \times \Omega \longrightarrow \mathbb{R}_{\geq 0}$ is a distance. If Ω, Ω' are properly convex subsets of $\mathbb{RP}^n$ and if $f : \Omega \longrightarrow \Omega'$ is the restriction of a projective map, then $d_{\Omega'}(f(x), f(y)) \leq d_\Omega(x,y)$. In particular every projective automorphism $f : \Omega \longrightarrow \Omega$ is an isometry. Every segment of projective line is a geodesic for this metric.

If Ω is strictly convex, the Hilbert metric is Finslerian, the geodesics are precisely the segments of projective lines, and the balls are strictly convex.

For a reference on the following facts see [Ki01, sect. 7]. Let $M = \Omega/\Gamma$ be a strictly convex projective manifold. The quotient M inherits a Finslerian metric from Ω. If g is a closed geodesic in M, its lift $\widetilde{g}$ is a segment of projective line, and it is invariant for an element $\gamma \in \Gamma$. The element γ acts on g as a translation of length ℓ_γ. The endpoints of $\widetilde{g}$ in $\partial\Omega$ are precisely the attracting and repelling fixed points of γ, x_γ and y_γ. The translation length ℓ_γ can be calculated from the eigenvalues λ_1 and λ_{n+1} as

$$\ell_\gamma = \log_e\left(\frac{\lambda_1}{\lambda_n}\right)$$

The function $\ell : \Gamma \longrightarrow \mathbb{R}_{>0}$ is called the **marked length spectrum** of M. The marked length spectrum determines the marked projective structure on M, see [Ki01, thm. 2].

2.3.4 Spaces of convex projective structures

Let M be a closed n-manifold such that the fundamental group $\pi_1(M)$ has trivial virtual center, it is Gromov hyperbolic, and it is torsion free. In particular M is orientable, as a non-orientable manifold has an element of the fundamental group of order 2.

For example every closed hyperbolic n-manifold whose fundamental group is torsion-free satisfies the hypotheses.

As usual we denote by $\mathcal{D}_{\mathbb{RP}^n}(M)$ and $\mathcal{T}_{\mathbb{RP}^n}(M)$ the spaces of based and marked projective structures on M.

We denote by $\mathcal{D}^c_{\mathbb{RP}^n}(M) \subset \mathcal{D}_{\mathbb{RP}^n}(M)$ and $\mathcal{T}^c_{\mathbb{RP}^n}(M) \subset \mathcal{T}_{\mathbb{RP}^n}(M)$ the subsets corresponding to convex projective structures on M, that are automatically strictly convex as $\pi_1(M)$ is Gromov hyperbolic.

Proposition 28. **[Koskul openness theorem]** *The subsets $\mathcal{D}^c_{\mathbb{RP}^n}(M) \subset \mathcal{D}_{\mathbb{RP}^n}(M)$ and $\mathcal{T}^c_{\mathbb{RP}^n}(M) \subset \mathcal{T}_{\mathbb{RP}^n}(M)$ are open.*

Proof : See [Ki01, sez. 6]. □

Proposition 29. *The holonomy map restricted to $\mathcal{D}^c_{\mathbb{RP}^n}(M)$ and $\mathcal{T}^c_{\mathbb{RP}^n}(M)$ is injective, identifying these spaces with their image, an open subset of* $\mathrm{Hom}(\pi_1(M), PGL_{n+1}(\mathbb{R}))$ *and* $\mathrm{Hom}(\pi_1(M), PGL_{n+1}(\mathbb{R}))/PGL_{n+1}(\mathbb{R})$ *respectively.*

Proof : By proposition 26, a strictly convex projective manifold Ω/Γ is determined by the group Γ, hence the holonomy is injective. □

Proposition 30. *The image of the map*

$$\pi_* : \mathrm{Hom}(\pi_1(M), SL^{\pm}_{n+1}(\mathbb{R})) \longrightarrow \mathrm{Hom}(\pi_1(M), PGL_{n+1}(\mathbb{R}))$$

contains the deformation space $\mathcal{D}^c_{\mathbb{RP}^n}(M)$. This map has a canonical section, identifying $\mathcal{D}^c_{\mathbb{RP}^n}(M)$ with an open subset of $\mathrm{Hom}(\pi_1(M), SL^{\pm}_{n+1}(\mathbb{R}))$.

Proof : By corollary 25, every element of $\mathcal{D}^c_{\mathbb{RP}^n}(M)$ has a canonical lift to $\mathrm{Hom}(\pi_1(M), SL^{\pm}_{n+1}(\mathbb{R}))$. □

Theorem 31. *With the identification of the previous proposition, $\mathcal{D}^c_{\mathbb{RP}^n}(M)$ is a finite union of connected components of* $\mathrm{Hom}(\pi_1(M), SL_{n+1}(\mathbb{R}))$, *in particular it is a clopen semi-algebraic subset.*

Proof : As M is orientable, all the representations corresponding to convex structures on M takes their values in $SL_{n+1}(\mathbb{R})$, hence $\mathcal{D}^c_{\mathbb{RP}^n}(M)$ is an open subset of $\mathrm{Hom}(\pi_1(M), SL_{n+1}(\mathbb{R}))$.

Given a group Γ, let

$$\mathcal{F}_\Gamma = \{\rho \in \mathrm{Hom}(\Gamma, SL_{n+1}(\mathbb{R})) \mid \rho \text{ is discrete and faithful and such that}$$

$$\rho(\Gamma) \text{ divides a properly convex open set } \Omega \subset \mathbb{RP}^m\}$$

Note that

$$\mathcal{F}_{\pi_1(M)} = \bigcup_N \mathcal{D}^c_{\mathbb{RP}^n}(N)$$

where N varies among all closed n-manifold with $\pi_1(N) = \pi_1(M)$. Hence $\mathcal{F}_{\pi_1(M)}$ is open in $\mathrm{Hom}(\pi_1(M), SL_{n+1}(\mathbb{R}))$.

A theorem of Benoist (see [Be3, thm. 1.1]) states that the set $\mathcal{F}_{\pi_1(M)}$ is also closed in $\mathrm{Hom}(\pi_1(M), SL_{n+1}(\mathbb{R}))$, hence it is a union of connected components of $\mathrm{Hom}(\pi_1(M), SL_{n+1}(\mathbb{R}))$. As $\mathcal{F}_{\pi_1(M)}$ is the disjoint union of the open sets $\mathcal{D}^c_{\mathbb{RP}^n}(N)$, each of them is also closed, in particular $\mathcal{D}^c_{\mathbb{RP}^n}(M)$ is a union of connected components of $\mathrm{Hom}(\pi_1(M), SL_{n+1}(\mathbb{R}))$.

As $\mathrm{Hom}(\pi_1(M), SL_{n+1}(\mathbb{R}))$ is an affine algebraic set, it has a finite number of connected components, and each of them is a semi-algebraic set. $\square$

Corollary 32. *The set of closed n-manifolds N with $\pi_1(N) = \pi_1(M)$ admitting a convex projective structure is finite.*

Proof : It follows from the proof of the theorem. $\square$

Now consider the semi-geometric quotient of $\mathrm{Hom}(\pi_1(M), SL_{n+1}(\mathbb{R}))$, and its image in the character variety

$$t : \mathrm{Hom}(\pi_1(M), SL_{n+1}(\mathbb{R})) \longrightarrow \overline{\mathrm{Char}}(\pi_1(M), SL_{n+1}(\mathbb{R}))$$

Proposition 33. *The image $t(\mathcal{D}^c_{\mathbb{RP}^n}(M))$ can be identified with the space $\mathcal{T}^c_{\mathbb{RP}^n}(M)$, it is a finite union of connected components (and, in particular, a clopen semi-algebraic subset) of $\overline{\mathrm{Char}}(\pi_1(M), SL_{n+1}(\mathbb{R}))$.*

Proof : By proposition 27, all representations in $\mathcal{D}^c_{\mathbb{RP}^n}(M)$ are absolutely irreducible, hence the image $t(\mathcal{D}^c_{\mathbb{RP}^n}(M))$ can be identified with $\mathcal{D}^c_{\mathbb{RP}^n}(M)/PGL_{n+1} = \mathcal{T}^c_{\mathbb{RP}^n}(M)$.

The clopen set $\mathcal{D}^c_{\mathbb{RP}^n}(M)$ is invariant for the action by conjugation of $PGL_{n+1}(\mathbb{R})$, hence by corollary 12 its image $\mathcal{T}^c_{\mathbb{RP}^n}(M)$ is clopen in $\overline{\mathrm{Char}}(\pi_1(M), SL_{n+1}(\mathbb{R}))$, in particular it is a semi-algebraic set. $\square$

Now we present a result showing that the space $\mathcal{T}^c_{\mathbb{RP}^n}(M)$ is often big enough to be interesting, as there are cases where we know a lower bound on the dimension of this space.

Proposition 34. *Suppose that M is a closed hyperbolic n-manifold containing r two-sided disjoint connected totally geodesic hypersurfaces. Then $\dim \mathcal{T}^c_{\mathbb{RP}^n}(M) \geq r$.*

Proof : The space $\mathcal{T}_{\mathbb{RP}^n}(M)$ has a special point x_0 corresponding to the hyperbolic structure. In [JM87] it is proven that in this case $\mathcal{T}_{\mathbb{RP}^n}(M)$ contains a ball of dimension r around x_0. As $\mathcal{T}^c_{\mathbb{RP}^n}(M)$ is open in $\mathcal{T}_{\mathbb{RP}^n}(M)$ and contains x_0, the dimension of $\mathcal{T}^c_{\mathbb{RP}^n}(M)$ is at least r. $\square$

Chapter 3

Logarithmic limit sets of real semi-algebraic sets

Logarithmic limit sets of complex algebraic sets have been extensively studied. They first appeared in Bergman's paper [Be71], and then they were further studied by Bieri and Groves in [BG81]. Recently their relations with the theory of non-archimedean fields and tropical geometry were discovered (see for example [SS04], [EKL06] and [BJSST07]). They are now usually called tropical varieties, but they appeared also under the names of Bergman fans, Bergman sets, Bieri-Groves sets or non-archimedean amoebas. The logarithmic limit set of a complex algebraic set is a polyhedral complex of the same dimension as the algebraic set, it is described by tropical equations and it is the image, under the component-wise valuation map, of an algebraic set over an algebraically closed non-archimedean field. The tools used to prove these facts are mainly algebraic and combinatorial.

In this chapter we extend these results to the logarithmic limit sets of real algebraic and semi-algebraic sets. The techniques we use to prove these results in the real case are very different from the ones used in the complex case. Our main tool is the cell decomposition theorem, as we prefer to look directly at the geometric set, instead of using its equations. In the real case, even if we restrict our attention to an algebraic set, it seems that the algebraic and combinatorial properties of the defining equations don't give enough information to study the logarithmic limit set.

In the following we often need to act on $(\mathbb{R}_{>0})^n$ with maps of the form:

$$\phi_A(x_1, \ldots, x_n) = (x_1^{a_{11}} \cdots x_n^{a_{1n}}, x_1^{a_{21}} \cdots x_n^{a_{2n}}, \ldots, x_1^{a_{n1}} \cdots x_n^{a_{nn}})$$

where $A = (a_{ij})$ is an $n \times n$ matrix. When the entries of A are not rational, the image of a semi-algebraic set is, in general not semi-algebraic. Actually, the only thing we can say about images of semi-algebraic sets via these maps is that they are definable in the structure of the real field expanded with arbitrary power functions. This structure, usually denoted by $\overline{\mathbb{R}}^{\mathbb{R}}$, is o-minimal and polynomially bounded, and these are the main properties we need in the proofs. Moreover, if S is a set definable in $\overline{\mathbb{R}}^{\mathbb{R}}$, then the image $\phi_A(S)$ is again definable, as the functions $x \longrightarrow x^\alpha$ are definable. This property is equivalent to say that $\overline{\mathbb{R}}^{\mathbb{R}}$ has field of exponents $\mathbb{R}$.

In this sense the category of semi-algebraic sets is too small for our methods. It seems that the natural context for the study of logarithmic limit sets is to fix a general expansion of the structure of the real field that is o-minimal and polynomially bounded with field of exponents $\mathbb{R}$. For sets definable in such a structure, the properties that were known for the complex algebraic sets also hold. We can prove that these logarithmic limit sets are polyhedral complexes with dimension less than or equal to the dimension of the definable set, and they are the image, under the component-wise valuation map, of an extension of the definable set to a real closed non-archimedean field. An analysis of the defining equations and inequalities is carried out, showing that the logarithmic limit set of a closed semi-algebraic set can be described applying the Maslov dequantization to a suitable formula defining the semi-algebraic set.

Our motivation for this work comes from the study of Teichmüller spaces and, more generally, of spaces of geometric structures on manifolds. In the next chapters we present a general construction of compactification using the logarithmic limit sets. The properties of logarithmic limit sets we prove here will be used in chapter 5 to describe the compactification. For example, the fact that logarithmic limit sets of real semi-algebraic sets are polyhedral complexes will provide an independent construction of the piecewise linear structure on the Thurston boundary of Teichmüller spaces. Moreover the relations with tropical geometry and the theory of non-archimedean fields will be used in the last chapter for constructing a geometric interpretation of the boundary points.

A brief description of the following sections. In section 3.1 we define a notion of logarithmic limit sets for general subsets of $(\mathbb{R}_{>0})^n$, and we report some preliminary notions of model theory and o-minimal geometry that we will use in the following, most notably the notion of regular polynomially bounded structures.

In section 3.2 we prove that logarithmic limit sets of definable sets in a regular polynomially bounded structure are polyhedral complexes with dimension less than or equal to the dimension of the definable set, and we provide a local description of these sets. The main tool we use in this section is the cell decomposition theorem.

In section 3.3 we show how the construction of Maslov dequantization provide a relation between logarithmic limit sets of semi-algebraic sets and tropical geometry.

3.1 Preliminaries

3.1.1 Some notations

If $x \in \mathbb{R}^n$ we will denote its coordinates by $x_1, \dots, x_n$. If $\omega \in \mathbb{N}^n$ we will use the multi-index notation for powers: $x^\omega = x_1^{\omega_1} \dots x_n^{\omega_n}$. We will consider also powers with real exponents, if the base is positive, hence if $x \in (\mathbb{R}_{>0})^n$ and $\omega \in \mathbb{R}^n$ we will write $x^\omega = x_1^{\omega_1} \dots x_n^{\omega_n}$.

If $s(k)$ is a sequence in $\mathbb{R}^n$, we will denote its k-th element by $s(k) \in \mathbb{R}^n$, and the coordinates by $s_i(k) \in \mathbb{R}$. We will not use the subscript notation for the indices of the sequences to avoid confusion with the indices of the coordinates.

Given a real number $\alpha > 1$, we will denote by Log_α the component-wise **logarithm map**:

$$\mathrm{Log}_\alpha : (\mathbb{R}_{>0})^n \ni (x_1, \dots, x_n) \longrightarrow (\log_\alpha(x_1), \dots, \log_\alpha(x_n)) \in \mathbb{R}^n$$

This map is analytic and bijective, and is inverse will be denoted by Exp_α:

$$\mathrm{Exp}_\alpha : \mathbb{R}^n \ni (x_1, \dots, x_n) \longrightarrow (\alpha^{x_1}, \dots, \alpha^{x_n}) \in (\mathbb{R}_{>0})^n$$

We define a notion of **limit** for every one-parameter family of subsets of $\mathbb{R}^n$. Suppose that for all $t \in (0, \varepsilon)$ we have a set $S_t \subset \mathbb{R}^n$. We can construct the deformation

$$\mathcal{D}(S_\cdot) = \{(x, t) \in \mathbb{R}^n \times (0, \varepsilon) \mid x \in S_t\}$$

We denote by $\overline{\mathcal{D}(S)}$ the closure of $\mathcal{D}(S)$ in $\mathbb{R}^n \times [0, \varepsilon)$, then we define

$$\lim_{t \to 0} S_t = \pi(\overline{\mathcal{D}(S)} \cap \mathbb{R}^n \times \{0\}) \subset \mathbb{R}^n$$

where $\pi : \mathbb{R}^n \times [0, \varepsilon) \longrightarrow \mathbb{R}^n$ is the projection on the first factor.

This kind of limit is well defined for every one parameter family of subsets of $\mathbb{R}^n$.

Proposition 35. *The set $S = \lim_{t \to 0} S_t$ is a closed subset of $\mathbb{R}^n$. A point y is in S if and only if there exist a sequence $y(k)$ in $\mathbb{R}^n$ and a sequence $t(k)$ in $(0, \varepsilon)$ such that $t(k) \to 0$, $y(k) \to x$ and $\forall k \in \mathbb{N} : y(k) \in S_{t(k)}$.* □

3.1.2 Logarithmic limit sets of general sets

Given a set $V \subset (\mathbb{R}_{>0})^n$ and a number $t \in (0, 1)$, we define the **amoeba** of V as

$$\mathcal{A}_t(V) = \mathrm{Log}_{\left(\frac{1}{t}\right)}(V) = \frac{-1}{\log_e(t)} \mathrm{Log}_e(V) \subset \mathbb{R}^n$$

Then we can define the **logarithmic limit set** of V as the limit of the amoebas:

$$\mathcal{A}_0(V) = \lim_{t \to 0} \mathcal{A}_t(V)$$

Some examples of logarithmic limit sets are in figures 3.1 and 3.2.

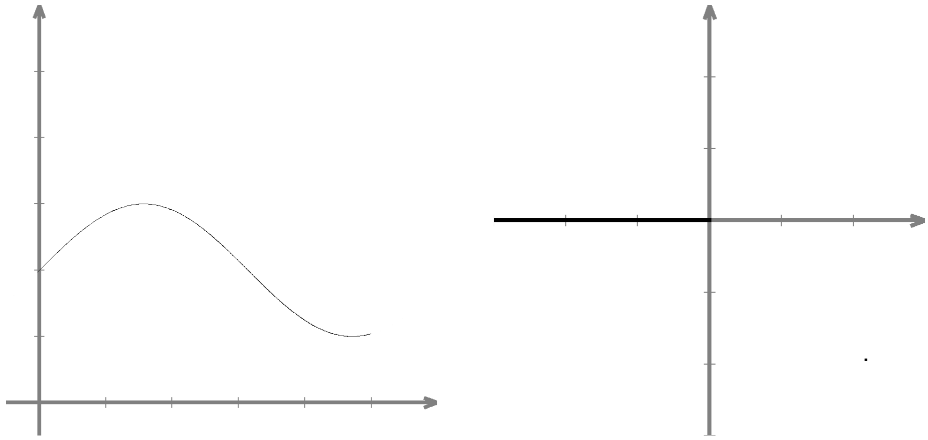

Figure 3.1: $V = \{(x,y) \in (\mathbb{R}_{>0})^2 \mid y = \sin x + 2, x \leq 5\}$ (left picture), then $\mathcal{A}_0(V) = \{(x,y) \in \mathbb{R}^2 \mid y = 0, x \leq 0\}$ (right picture).

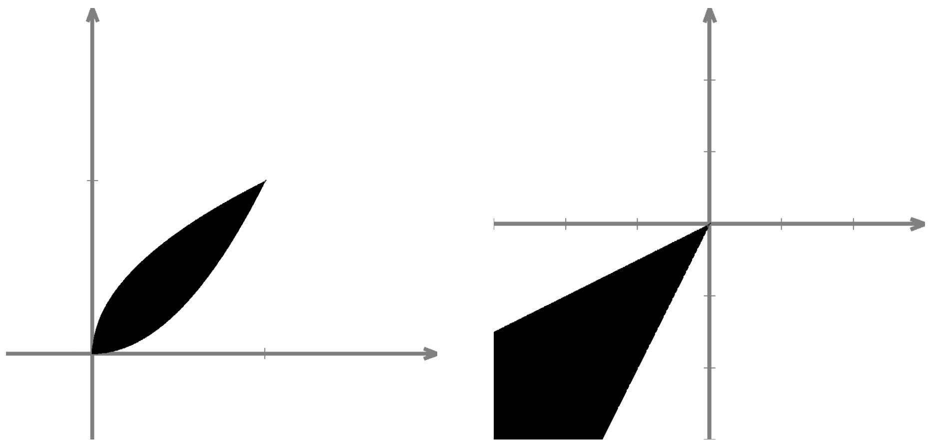

Figure 3.2: $V = \{(x,y) \in (\mathbb{R}_{>0})^2 \mid x^2 \leq y \leq \sqrt{x}\}$ (left picture), then $\mathcal{A}_0(V) = \{(x,y) \in \mathbb{R}^2 \mid 2x \leq y \leq \frac{1}{2}x\}$ (right picture).

Proposition 36. *Given a set $V \subset (\mathbb{R}_{>0})^n$ the following properties hold:*

1. *The logarithmic limit set $\mathcal{A}_0(V)$ is closed and $y \in \mathcal{A}_0(V)$ if and only if there exist a sequence $x(k)$ in V, and a sequence $t(k)$ in $(0,1)$ such that $t(k) \to 0$ and*
$$\mathrm{Log}_{\left(\frac{1}{t(k)}\right)}(x(k)) \to y$$
2. *The logarithmic limit set $\mathcal{A}_0(V)$ is a cone in $\mathbb{R}^n$.*
3. *We have that $0 \in \mathcal{A}_0(V)$ if and only if $V \neq \emptyset$. Moreover, $\mathcal{A}_0(V) = \{0\}$ if and only if V is compact and non-empty.*
4. *If $W \subset \mathbb{R}^n$ we have $\mathcal{A}_0(V \cup W) = \mathcal{A}_0(V) \cup \mathcal{A}_0(W)$ and $\mathcal{A}_0(V \cap W) \subset \mathcal{A}_0(V) \cap \mathcal{A}_0(W)$.*

Proof : The first assertion is simply a restatement of proposition 35.

For the second one, we want to prove that if $\lambda > 0$ and $y \in \mathcal{A}_0(V)$, then $\lambda^{-1}y \in \mathcal{A}_0(V)$. There exists a sequence $x(k)$ in V and a sequence $t(k)$ in $(0,1)$ such that $t(k) \to 0$ and $\mathrm{Log}_{\left(\frac{1}{t(k)}\right)}(x(k)) \to y$. Consider the sequence $x(k)$ and the sequence $t(k)^\lambda$. Now

$$\mathrm{Log}_{\left(\frac{1}{t(k)^\lambda}\right)}(x(k)) = \frac{-1}{\log_e(t(k)^\lambda)}\mathrm{Log}_e(x(k)) = \lambda^{-1}\frac{-1}{\log_e(t(k))}\mathrm{Log}_e(x(k))$$

and this sequence converges to $\lambda^{-1}y$.

The third and fourth assertions are trivial. □

Given a closed cone $C \subset \mathbb{R}^n$, there is always a set $V \subset (\mathbb{R}_{>0})^n$ such that $C = \mathcal{A}_0(V)$, simply take $V = \mathrm{Log}_e^{-1}(C)$. Then $\mathcal{A}_t(V)$ does not depend on t, and it is equal to C.

Let $A = (a_{ij}) \in GL_n(\mathbb{R})$. The matrix A acts on $\mathbb{R}^n$ in the natural way and, via conjugation with the map Log_e, it acts on $(\mathbb{R}_{>0})^n$. Explicitly, it induces the maps $A : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ and $\overline{A} : (\mathbb{R}_{>0})^n \longrightarrow (\mathbb{R}_{>0})^n$:

$$A(x) = A(x_1, \dots, x_n) = (a_{11}x_1 + \dots + a_{1n}x_n,\ \dots\ , a_{n1}x_1 + \dots + a_{nn}x_n)$$

$$\overline{A}(x) = \mathrm{Exp}_e \circ A \circ \mathrm{Log}_e(x) = (x_1^{a_{11}}x_2^{a_{12}} \cdots x_n^{a_{1n}}\ ,\ \dots\ ,\ x_1^{a_{n1}}x_2^{a_{n2}} \cdots x_n^{a_{nn}})$$

If $V \subset (\mathbb{R}_{>0})^n$ and $B \in GL_n(\mathbb{R})$, then $B(\mathcal{A}_0(V))$ is the logarithmic limit set of $\overline{B}(V)$.

Lemma 37. $(0, \dots, 0, -1) \in \mathcal{A}_0(V)$ *if and only if there exists a sequence $y(k)$ in V such that $y_n(k) \longrightarrow 0$ and*

$$\forall N \in \mathbb{N} : \exists k_0 \in \mathbb{N} : \forall k > k_0 : \forall i \in \{1, \dots, n-1\} :$$
$$y_n(k) < (y_i(k))^N \text{ and } y_n(k) < (y_i(k))^{-N}$$

Proof : Suppose that $(0,\dots,0,-1) \in \mathcal{A}_0(V)$, then there exist a sequence $y(k)$ in V and a sequence $t(k)$ in $(0,1)$ such that $t(k) \to 0$ and $\mathrm{Log}_{\left(\frac{1}{t(k)}\right)}(y(k)) \to (0,\dots,0,-1)$. This means that

$$\frac{-1}{\log_e(t(k))}\log_e(y_n(k)) \to \; -1$$

and

$$\forall i \in \{1,\dots,n-1\} : \frac{-1}{\log_e(t(k))}\log_e(y_i(k)) \to 0$$

As $t(k) \to 0$, then $\frac{-1}{\log_e(t(k))} \to 0^+$, $\log_e(y_n(k)) \to -\infty$ and $y_n(k) \to 0$. Moreover

$$\forall i \in \{1,\dots,n-1\} : \frac{\log_e(y_i(k))}{\log_e(y_n(k))} \to 0$$

Hence $\forall \varepsilon > 0 : \exists k_0 : \forall k > k_0 : \forall i \in \{1,\dots,n-1\}$:

$$\left|\frac{\log_e(y_i(k))}{\log_e(y_n(k))}\right| < \varepsilon$$

$$(y_n(k))^{\varepsilon} < y_i(k) < (y_n(k))^{-\varepsilon}$$

We conclude by reversing the inequalities and choosing $\varepsilon = \frac{1}{N}$.

Conversely, if $y(k)$ has the stated property, then $|\,\mathrm{Log}_e(y(k))| \to \infty$. It is possible to choose $t(k)$ such that $t(k) \to 0$ and $\left|\mathrm{Log}_{\left(\frac{1}{t(k)}\right)}(y(k))\right| = 1$. Up to subsequences, the sequence $\mathrm{Log}_{\left(\frac{1}{t(k)}\right)}(y(k))$ converges to a point that, by reversing the calculations on first part of the proof, is $(0,\dots,0,-1)$. Hence $(0,\dots,0,-1) \in \mathcal{A}_0(V)$. □

Corollary 38. *It follows that if there exists a sequence $x(k)$ in V such that $x(k) \to (a_1,\dots,a_{n-1},0)$, where $a_1,\dots,a_{n-1} > 0$, then $(0,\dots,0,-1) \in \mathcal{A}_0(V)$. The converse is not true in general.*

Proof : For the counterexample, see figure 3.3. □

We will see in theorem 47 that if V is definable in an o-minimal and polynomially bounded structure, the converse of the corollary becomes true.

A sequence $b(k)$ in $(\mathbb{R}_{>0})^n$ is in **standard position** in dimension m if, denoted $g = n-m$, $b(k) \to b = (b_1,\dots,b_g,0,\dots,0)$, with $b_1,\dots,b_g > 0$ and:

$$\forall N \in \mathbb{N} : \exists k_0 : \forall k > k_0 : \forall i \in \{g+1,\dots,n-1\} : b_{i+1}(k) < (b_i(k))^N$$

Lemma 39. *Let $a(k)$ be a sequence in $(\mathbb{R}_{>0})^n$ such that $a(k) \to a = (a_1,\dots,a_h,0,\dots,0)$, with $h < n$ and $a_1,\dots,a_h > 0$. There exists a subsequence (again denoted by $a(k)$) and a linear map $A : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ such that*

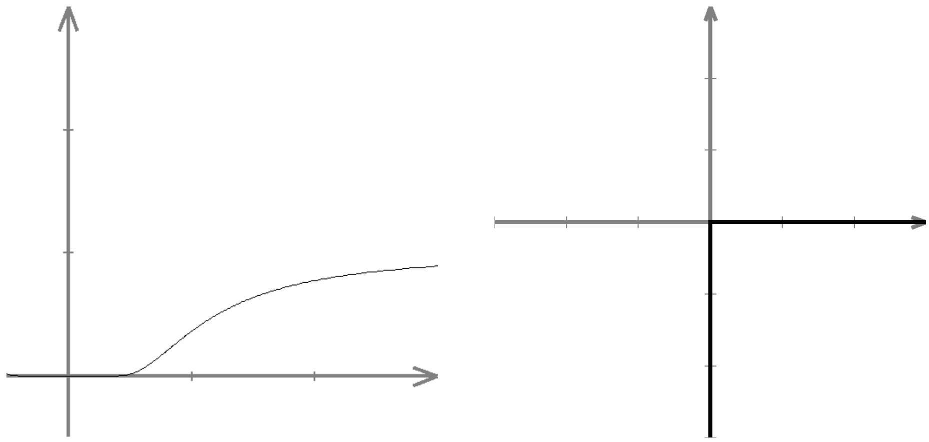

Figure 3.3: $V = \{(x,y) \in (\mathbb{R}_{>0})^2 \mid y = e^{-\frac{1}{x^2}}\}$ (left picture), then $\mathcal{A}_0(V) = \{(x,y) \in \mathbb{R}^2 \mid y = 0, x \geq 0 \text{ or } x = 0, y \leq 0\}$ (right picture).

the sequence $b(k) = \left(\overline{A}(a(k))\right) \subset (\mathbb{R}_{>0})^n$ *is in standard position in dimension* m*, with* $g = n - m \geq h$*.*

Proof : By induction on n. For $n = 1$ the statement is trivial. Suppose that the statement holds for $n-1$. Consider the logarithmic image of the sequence: $\mathrm{Log}(a(k))$. Up to extracting a subsequence, the sequence $\left(\frac{\mathrm{Log}(a(k))}{|\mathrm{Log}(a(k))|}\right)$ converges to a unit vector $v = (0, \ldots, 0, v_{h+1}, \ldots, v_n)$. There exists a linear map B, acting only on the last $n-h$ coordinates, sending v to $(0, \ldots, 0, -1)$. By lemma 37, the map $\overline{B}$ sends $a(k)$ to a sequence $b(k)$ such that $b_n(k) \to 0$ and

$$\forall N \in \mathbb{N} : \exists k_0 : \forall k > k_0 : \forall i \in \{1, \ldots, n-1\} :$$

$$b_n(k) < (b_i(k))^N \text{ and } b_n(k) < (b_i(k))^{-N}$$

As B only acted on the last $n-h$ coordinates, for $i \in \{1, \ldots, h\}$, $b_i(k) \to a_i \neq 0$. Up to subsequences we can suppose that for every $i \in \{h+1, \ldots, n-1\}$ one of the three possibilities occur: $b_i(k) \to 0$, $b_i(k) \to b_i \neq 0$, $b_i(k) \to +\infty$. Up to a change of coordinates with maps of the form

$$B_i(x_1, \ldots, x_i, \ldots, x_n) = (x_1, \ldots, -x_i, \ldots, x_n)$$

$$\overline{B_i}(x_1, \ldots, x_i, \ldots, x_n) = (x_1, \ldots, x_i^{-1}, \ldots, x_n)$$

we can suppose that for every $i \in \{1, \ldots, n-1\}$ either $b_i(k) \to 0$ or $b_i(k) \to b_i \neq 0$. Up to reordering the coordinates, we can suppose that exists $g \geq h$ such that for $i \in \{1, \ldots, g\}$: $b_i(k) \to b_i \neq 0$ and for $i > g$, $b_i(k) \to 0$. Now consider the projection on the first $n-1$ coordinates: $\pi : \mathbb{R}^n \longrightarrow \mathbb{R}^{n-1}$.

By inductive hypothesis there exists a linear map $C : \mathbb{R}^{n-1} \longrightarrow \mathbb{R}^{n-1}$ sending the sequence $\pi(b(k))$ in a sequence $c(k)$ satisfying:

$$\forall N \in \mathbb{N} : \exists k_0 : \forall k > k_0 : \forall i \in \{g+1, \dots, n-2\} : b_{i+1}(k) < (b_i(k))^N$$

The composition of B and a map that preserves the last coordinate and acts as C on the first ones is the searched map. $\square$

The **basic cone** in $\mathbb{R}^n$ defined by the vector $N = (N_1, \dots, N_{n-1}) \in \mathbb{N}^{n-1}$ is the set:

$$B_N = \{x \in \mathbb{R}^n \mid \forall i : x_i \leq 0 \text{ and } \forall i < n : x_{i+1} \leq N_i x_i\}$$

Note that if $N' = (N'_1, \dots, N'_{n-1})$, with $\forall i : N'_i \geq N_i$, then $B'_N \subset B_N$.

The **exponential basic cone** in $(\mathbb{R}_{>0})^n$ defined by the vector $N = (N_1, \dots, N_{n-1}) \in \mathbb{N}^{n-1}$ and the scalar $h > 0$ is the set:

$$E_{N,h} = \{x \in \mathbb{R}^n \mid \forall i : 0 < x_i \leq h \text{ and } \forall i < n : x_{i+1} \leq {x_i}^{N_i}\}$$

Lemma 40. *The following easy facts about basic cones holds:*

1. *The logarithmic limit set of an exponential basic cone is a basic cone:*

$$\mathcal{A}_0(E_{N,h}) = B_N$$

2. *If $b(k) \subset (\mathbb{R}_{>0})^n$ is a sequence in standard position in dimension n, and $E_{N,h}$ is an exponential basic cone, then for every sufficiently large k, $b(k) \in E_{N,h}$.*

$\square$

3.1.3 Definable sets in o-minimal structures

In this subsection we report some definitions of model theory and o-minimal geometry we will use later, see [EFT84] and [Dr] for details. A **set of symbols** S is a triple $S = (\mathcal{R}, \mathcal{F}, \mathcal{C})$, where every element of $\mathcal{R}$ is an n-ary **relation symbol** for some $n \geq 1$, every element of $\mathcal{F}$ is an n-ary **function symbol** for some $n \geq 1$ and every element of $\mathcal{C}$ is a **constant symbol** ([EFT84, chap. II, def. 2.1]). A set of symbols $S' = (\mathcal{R}', \mathcal{F}', \mathcal{C}')$ is an **expansion** of S if $\mathcal{R}' \subset \mathcal{R}$, $\mathcal{F}' \subset \mathcal{F}$, $\mathcal{C}' \subset \mathcal{C}$. The theory of real closed fields uses the set of symbols of ordered semirings: $\mathcal{OS} = (\{\leq\}, \{+, \cdot\}, \emptyset)$ or, equivalently, the set of symbols of ordered rings $\mathcal{OR} = (\{\leq\}, \{+, -, \cdot\}, \{0, 1\})$, an expansion of $\mathcal{OS}$. In the following we will use these sets of symbols and some of their expansions. Every set of symbols S defines a **first order language** L_S ([EFT84, chap. II, def. 3.2]).

Given a set of symbols $S = (\mathcal{R}, \mathcal{F}, \mathcal{C})$, an S-**structure** is a pair $\overline{M} = (M, a)$, where M is a set, and a is a map, called **interpretation**, defined on $\mathcal{R} \cup \mathcal{F} \cup \mathcal{C}$, such that for every n-ary relation symbol R, $a(R) \subset M^n$, for every n-ary function symbol f, $a(f) : M^n \longrightarrow M$, for every constant symbol c, $a(c) \in M$ (see [EFT84, cap. III, def. 1.1]).

Given an S-structure $\overline{M} = (M, a)$, and an L_S-sentence ϕ (a formula without free variables), we will write $\overline{M} \vDash \phi$ if $\overline{M}$ satisfies ϕ (see [EFT84, chap. III, def. 3.1]). Two S-structures $\overline{M}$ and $\overline{N}$ are **elementary equivalent** if for all L_S-sentences ϕ:

$$\overline{M} \vDash \phi \Leftrightarrow \overline{N} \vDash \phi$$

A real closed field can be defined as an $\mathcal{OS}$- or an $\mathcal{OR}$-structure satisfying a suitable infinite set of first order axioms. The natural $\mathcal{OS}$-structure on $\mathbb{R}$ will be denoted by $\overline{\mathbb{R}}$. Two real closed fields are elementary equivalent $\mathcal{OS}$-structures.

If $\overline{M} = (M, a)$ is an S-structure, and S' is an expansion of S, an S'-structure (M, a') is an **expansion** of the S-structure (M, a) if a' restricted to the symbols of S is equal to a. For example, for every set of functions F (every element of F is a function $M^n \longrightarrow M$ for some n), we can construct an expansion S' of S by adding a function symbol for every element of F, and we can construct an expansion (M, a') of the S-structure (M, a) by interpreting every function symbol with the corresponding element of F.

If $M \subset N$, an S-structure $\overline{N} = (N, b)$ is an **extension** of an S-structure $\overline{M} = (M, a)$ if for all $s \in \mathcal{R} \cup \mathcal{F} \cup \mathcal{C}$, $b(s)_{|M} = a(s)$. The S-structure $\overline{N}$ is called an **elementary extension** of the S-structure $\overline{M}$ if chosen an L_S-formula $\phi(x_1, \ldots, x_n)$ and parameters $a_1, \ldots, a_n \in M$, then

$$\overline{N} \vDash \phi(a_1, \ldots, a_n) \Leftrightarrow \overline{M} \vDash \phi(a_1, \ldots, a_n)$$

In particular if $\overline{N}$ is an elementary extension of $\overline{M}$, they are elementarily equivalent.

If $\overline{M}$ is an S-structure, given an (L_S)-formula $\phi(x_1, \ldots, x_n, y_1, \ldots, y_m)$ with free variables $x_1, \ldots, x_n, y_1, \ldots, y_m$, and given parameters $a_1, \ldots a_m \in M$, the set

$$\{x \in M^n \mid \phi(x_1, \ldots, x_n, a_1, \ldots, a_m)\}$$

is called a **definable set** (in the S-structure (M, a)). For every n there is a well defined class D_n of definable subsets of M^n. The classes of definable sets are closed by union, intersection, complement, projection, inverse image by projection and Cartesian product.

Given $A \subset M^n$, a map $f : A \longrightarrow M^m$ is called **definable** if its graph is a definable subset of M^{n+m}. In this case A and $f(A)$ are definable. Composition of definable maps is definable, and if a definable map is injective, its inverse is definable.

For example if M is an $\mathcal{OS}$-structure satisfying the axioms of real closed fields, the definable sets are the semi-algebraic sets, and the definable maps are the semi-algebraic maps. Usually the definition of a semi-algebraic set is given using only quantifier-free formulae, in this case the equivalence follows from the Tarski-Seidenberg principle, see [BCR98, 2.2.4].

Let S be an expansion of $\mathcal{OS}$, and let $\overline{M}$ be an S-structure satisfying the axioms of the real closed fields. The class D_1 of definable subsets of M contains all finite unions of single points and open intervals (bounded and unbounded). If these are the only elements of D_1, then the S-structure $\overline{M}$ is called **o-minimal**.

Let S be an expansion of $\mathcal{OS}$, and let $\mathfrak{R} = (\mathbb{R}, a)$ be an S-structure that is an expansion of $\overline{\mathbb{R}}$. Moreover, suppose that $\mathfrak{R}$ is o-minimal. The structure $\mathfrak{R}$ is called **polynomially bounded** if for every definable function $f : \mathbb{R} \longrightarrow \mathbb{R}$ there exists a natural number N such that for every sufficiently large x, $|f(x)| \leq x^N$. In [Mi94] it is shown that if $\mathfrak{R}$ is polynomially bounded and $f : \mathbb{R} \longrightarrow \mathbb{R}$ is definable and not ultimately 0, there exist $r, c \in \mathbb{R}$, $c \neq 0$, such that the function $x \longrightarrow x^r : (0, +\infty) \longrightarrow (0, +\infty)$ is definable and

$$\lim_{x \to +\infty} \frac{f(x)}{x^r} = c$$

The set of all such r is a subfield of $\mathbb{R}$, called the **field of exponents** of $\mathfrak{R}$. For example the $\mathcal{OR}$-structure $\overline{\mathbb{R}}$ is polynomially bounded with field of exponents $\mathbb{Q}$.

If $\Lambda \subset \mathbb{R}$ is a subfield, and if the field of exponents of an S-structure $\mathfrak{R}$ does not contain Λ, we can construct an expansion of S and $\mathfrak{R}$ by adding the power functions with exponents in Λ. We expand S to S^Λ by adding a function symbol f_λ for every $\lambda \in \Lambda$, and we expand $\mathfrak{R}$ to an S^Λ-structure $\mathfrak{R}^\Lambda$ interpreting the function symbol f_λ by the function that is $x \longrightarrow x^\lambda$ for positive numbers and $x \longrightarrow 0$ on negative ones. The structure $\mathfrak{R}^\Lambda$ is again o-minimal, as its definable sets are definable in the structure $\mathfrak{R}$ expanded with the exponential function $x \longrightarrow e^x$, that is o-minimal by [Spe99].

It is not known whether $\mathfrak{R}^\Lambda$ is always polynomially bounded. The only theorem we now in this direction is the following: if the expansion of $\mathfrak{R}$ constructed by adding the family of functions $\{x^r_{|[1,2]}\}_{r \in \Lambda}$ is polynomially bounded, then $\mathfrak{R}^\Lambda$ is too (see [Mi03]). Hence if we want to prove that $\mathfrak{R}^\Lambda$ is polynomially bounded we only need to search for a polynomially bounded expansion of $\mathfrak{R}$ in which all restricted powers $x^r_{|[1,2]}$ are definable. For example if the structure $\mathfrak{R}$ expanded with the restricted exponential, $e^x_{|[0,1]}$ is polynomially bounded, then $\mathfrak{R}^\Lambda$ is too.

In the following we will work with o-minimal, polynomially bounded structures $\mathfrak{R}$ expanding $\overline{\mathbb{R}}$, with the property that $\mathfrak{R}^\mathbb{R}$ is polynomially bounded. We will call such structures **regular structures**.

Example 41. *Let's see some examples of regular polynomially bounded structures.*

1. *The structure $\mathbb{R}_{\mathrm{an}}$ of the real numbers with restricted analytic functions, an expansion of $\overline{\mathbb{R}}$ where for every n-variable power series f converging in a neighborhood of the unit cube $[-1,1]^n$ a new function symbol is added representing f on the unit cube and extended to zero outside the cube, see [DMM94] for details. This structure is o-minimal and polynomially bounded with field of exponents $\mathbb{Q}$, and all restricted powers $\{x^r_{|[1,2]}\}$ are definable, hence its expansion with all the power functions $\mathbb{R}_{\mathrm{an}}^{\mathbb{R}}$ is o-minimal, polynomially bounded with fields of exponents $\mathbb{R}$.*

2. *The structure $\overline{\mathbb{R}}^{\mathbb{R}}$ is o-minimal and polynomially bounded, as all definable sets in this structure are definable in $\mathbb{R}_{\mathrm{an}}^{\mathbb{R}}$. This is the smallest structure with field of exponents $\mathbb{R}$.*

3. *The structure $\mathbb{R}_{\mathrm{an}^*}$ of the real field with convergent generalized power series, see [DS98] for details of the definition, is o-minimal and polynomially bounded with field of exponents $\mathbb{R}$.*

4. *Other structures with field of exponents $\mathbb{Q}$ that stays polynomially bounded when expanded with the power functions are the field of real numbers with multisummable series (see [DS00]) and the structures defined by a quasianalytic Denjoy-Carlemann class (see [RSW03]).*

3.2 Logarithmic limit sets of definable sets

3.2.1 Some properties of definable sets

Let $\mathfrak{R}$ be an o-minimal and polynomially bounded expansion of $\overline{\mathbb{R}}$.

Lemma 42. *For every definable function $f : (\mathbb{R}_{>0})^n \longrightarrow \mathbb{R}_{>0}$, there is a basic exponential cone C and $N \in \mathbb{N}$ such that $f_{|C}(x_1, \dots, x_n) \geq (x_n)^N$.*

Proof : Fix a basic exponential cone $C \subset (\mathbb{R}_{>0})^n$. By the Łojasiewicz inequality (see [DM96, 4.14]) there exist $N \in \mathbb{N}$ and $Q > 0$ such that $Qf_{|C}(x_1, \dots, x_n) \geq (x_n)^N$. The thesis follows by choosing an exponent bigger than N and a suitable basic exponential cone smaller than C. □

Lemma 43. *Every cell decomposition of $(\mathbb{R}_{>0})^n$ has a cell containing a basic exponential cone.*

Proof : This proof is based on the cell decomposition theorem, see [Dr, chap. 3] for details. By induction on n. For $n = 1$, the statement is obvious, a basic exponential cone being an interval with 0 as an infimum.

Suppose the lemma true for n. If $\{C_i\}$ is a cell decomposition of $(\mathbb{R}_{>0})^{n+1}$, and if $\pi : (\mathbb{R}_{>0})^{n+1} \longrightarrow (\mathbb{R}_{>0})^n$ is the projection on the first n coordinates, then $\{\pi(C_i)\}$ is a cell decomposition of $(\mathbb{R}_{>0})^n$, hence, by induction, it contains a basic exponential cone D of $(\mathbb{R}_{>0})^n$. The cells in $\pi^{-1}(D) \cup (0,1]$ are described by functions

$$f_1, \dots, f_l : D \longrightarrow [0,1]$$

with $f_1 < f_2 < \cdots < f_n$ and such that every cell in $\pi^{-1}(D) \cup (0,1]$ has one of the forms:

$$\{(\bar{x}, x_{n+1}) \mid \bar{x} \in D, x_{n+1} = f_i(\bar{x})\}$$

$$\{(\bar{x}, x_{n+1}) \mid \bar{x} \in D, f_i(\bar{x}) < x_{n+1} < f_{i+1}(\bar{x})\}$$

where $\bar{x} = (x_1, \dots, x_n)$. Then f_1 is identically zero, while f_2 takes values in $\mathbb{R}_{>0}$, hence, by previous lemma, there is a basic exponential cone $D' \subset D$ and $N \in \mathbb{N}$ such that $f_{2|D'}(\bar{x}) \geq (x_n)^N$. Hence the piece $\{(\bar{x}, x_{n+1}) \mid \bar{x} \in D, f_1(\bar{x}) < x_{n+1} < f_2(\bar{x})\}$ contains the basic exponential cone

$$\{(\bar{x}, x_{n+1}) \mid \bar{x} \in D', 0 < x_{n+1} \leq (x_n)^N\}$$

□

Corollary 44. *Let $V \subset (\mathbb{R}_{>0})^n$ be definable in $\mathfrak{R}$, and suppose that V contains a sequence $x(k)$ in standard position in dimension n. Then V contains an exponential cone.*

Proof : Let $\{C_i\}$ be a cell decomposition of $(\mathbb{R}_{>0})^n$ adapted to V, i.e. every C_i is either contained in V or disjoint from V. By previous lemma, one of these cells contains an exponential cone D. By hypothesis, if k is sufficiently large, $x(k) \in D$, hence $D \subset V$. □

Corollary 45. *Let $V \subset (\mathbb{R}_{>0})^2$ be definable in $\mathfrak{R}$, and suppose that exists a sequence $x(k)$ in V such that $x(k) \to 0$ and*

$$\forall N \in \mathbb{N} : \exists k_0 : \forall k > k_0 : x_2(k) < (x_1(k))^N$$

Then there exist $h_0 > 0$ and $M \in \mathbb{N}$ such that

$$\{x \in \mathbb{R}^2 \mid 0 < x_1 < h_0 \text{ and } 0 < x_2 < (x_1)^M\} \subset V$$

Proof : This is precisely the previous corollary with $n = 2$. □

Lemma 46. *Let $V \subset (\mathbb{R}_{>0})^n$ be definable in $\mathfrak{R}$, and suppose that there exists a sequence $x(k)$ in V, an integer $m \in \{1, \dots, n\}$ and, denoted $g = n - m$, positive numbers $a_1, \dots, a_g > 0$ such that $x(k) \to (a_1, \dots, a_g, 0, \dots, 0)$, and such that:*

$$\forall N \in \mathbb{N} : \exists k_0 : \forall k > k_0 : \forall i \in \{g+1, \dots, n-1\} : x_n(k) < (x_i(k))^N$$

Then for every $\varepsilon > 0$ there exist a sequence $y(k)$ in V and positive real numbers $b_1, \dots b_{n-1} > 0$ such that $y(k) \to (b_1, \dots b_{n-1}, 0)$ and for all $i \in \{1, \dots g\}$ we have $|b_i - a_i| < \varepsilon$.

Proof : If $n = 2$ the statement follows by corollary 45. By induction on n we suppose the statement true for definable sets in $\mathbb{R}^{n'}$ with $n' < n$. We split the proof in two cases, when $m < n$ and when $m = n$.

If $m < n$, fix an $\varepsilon > 0$, smaller than any of the a_i, and consider the parallelepiped

$$c_\varepsilon = \{(z_1, \dots, z_n) \in \mathbb{R}^n \mid |z_1 - a_1| < \tfrac{1}{2}\varepsilon, \dots, |z_g - a_g| < \tfrac{1}{2}\varepsilon\}$$

Let $\pi : \mathbb{R}^n \longrightarrow \mathbb{R}^{n-m}$ be the projection on the last $n - m$ coordinates. The set $\pi(V \cap c_\varepsilon)$ is definable in $\mathbb{R}^m$, the sequence $\pi(x(k))$ satisfies the hypotheses of the lemma, hence, by induction, there exists a sequence $z(k) \in \pi(V \cap c_\varepsilon)$ converging to the point $(b_{g+1}, \dots, b_{n-1}, 0)$. Let $y(k)$ be a sequence such that $y(k) \in \pi^{-1}(z(k))$. We can extract a subsequence (called again $y(k)$) such that $y(k) \to (0, b_2, \dots b_n)$ where for all $i \in \{1, \dots g\}$ we have $|b_i - a_i| \leq \frac{1}{2}\varepsilon$.

If $m = n$, then $x(k) \to 0$. The sequence $\left(\frac{(x_1(k),\dots,x_{n-1}(k))}{|(x_1(k),\dots,x_{n-1}(k))|}\right)$ is contained in the unit sphere S^{n-2}, and, up to subsequences, we can suppose that it converges to a unit vector $v = (v_1, \dots, v_{n-1}) \in (\mathbb{R}_{\geq 0})^{n-1}$. Up to reordering, $v = (v_1, \dots, v_h, 0, \dots, 0)$, with $v_1, \dots, v_h > 0$. Fix an $\alpha > 0$ and consider the cone

$$C_v(\alpha) = \{y \in \mathbb{R}^n \mid \frac{\langle (y_1, \dots, y_{n-1}), v\rangle}{|(y_1, \dots, y_{n-1})||v|} > \cos\alpha\}$$

Let $\pi : \mathbb{R}^n \longrightarrow \mathbb{R}^{n-h}$ the projection on the last $n - h$ coordinates. The set $\pi(V \cap C_v(\alpha))$ is definable in $\mathbb{R}^{n-h}$, the sequence $\pi(x(k))$ satisfies the hypotheses of the lemma, hence, by induction, there exists a sequence $z(k) \in \pi(V \cap C_v(\alpha))$ converging to the point $(b_{h+1}, \dots, b_{n-1}, 0)$. Let $y(k)$ be a sequence such that $y(k) \in \pi^{-1}(z(k))$. We can extract a subsequence (called again $y(k)$) such that $y(k) \to (b_1, \dots b_{n-1}, 0)$. As $y(k) \in C_v(\alpha)$, for all $i > h$, if $y_i(k) \to 0$, then $y(k) \to 0$. As $b_{h+1}, \dots, b_{n-1} > 0$, then also $b_1, \dots, b_h > 0$. $\square$

3.2.2 Polyhedral structure

Let $V \subset (\mathbb{R}_{>0})^n$ be a definable set. Our main object of study is $\mathcal{A}_0(V)$, the logarithmic limit set of V. Suppose that $\mathfrak{R}$ has field of exponents $\Lambda \subset \mathbb{R}$. Given a matrix $B \in GL_n(\Lambda)$, the set $B(\mathcal{A}_0(V))$ is the logarithmic limit set of $\overline{B}(V)$. If

$$V = \{x \in (\mathbb{R}_{>0})^n \mid \phi(x_1, \dots, x_n, a_1, \dots, a_m)\}$$

is a definition of V, then

$$\overline{B}(V) = \{x \in (\mathbb{R}_{>0})^n \mid \phi(\overline{B^{-1}}(x_1), \dots, \overline{B^{-1}}(x_n), a_1, \dots, a_m)\}$$

The components of $\overline{B^{-1}}$ are all definable in $\mathfrak{R}$ because their exponents are in Λ, hence the set $\overline{B}(V)$ is again definable.

The group $GL_n(\Lambda)$ is transitive on Λ^n, hence we can always find a matrix sending a specific point of Λ^n in a point of our choice, like $(0, \dots, 0, -1)$.

Theorem 47. *Let $V \subset (\mathbb{R}_{>0})^n$ be a set definable in an o-minimal and polynomially bounded structure. The point $(0, \dots, 0, -1)$ is in $\mathcal{A}_0(V)$ if and only if there exists a sequence $x(k)$ in V such that $x(k) \to (a_1, \dots, a_{n-1}, 0)$, where $a_1, \dots, a_{n-1} > 0$.*

Proof: If there exists such an $x(k)$, then it is obvious that $(0, \dots, 0, -1) \in \mathcal{A}_0(V)$. Vice versa, if $(-1, 0, \dots, 0) \in \mathcal{A}_0(V)$, then by lemma 37 there exists a sequence $y(k)$ in V such that $y_n(k) \longrightarrow 0$ and

$$\forall N \in \mathbb{N} : \exists k_0 \in \mathbb{N} : \forall k > k_0 : \forall i \in \{1, \dots, n-1\} :$$
$$y_n(k) < (y_i(k))^N \text{ and } y_n(k) < (y_i(k))^{-N}$$

Up to subsequences we can suppose that for all $i \in \{1, \dots, n-1\}$ one of the three possibilities occur: $y_i(k) \to 0$, $y_i(k) \to a_i \neq 0$, $y_i(k) \to +\infty$. Up to a change of coordinates with maps of the form

$$B_i(x_1, \dots, x_i, \dots, x_n) = (x_1, \dots, -x_i, \dots, x_n)$$
$$\overline{B_i}(x_1, \dots, x_i, \dots, x_n) = (x_1, \dots, x_i^{-1}, \dots, x_n)$$

we can suppose that for all $i \in \{1, \dots, n-1\}$ either $y_i(k) \to 0$ or $y_i(k) \to a_i \neq 0$. Then we can apply lemma 46, and we are done. □

Proposition 48. *Let $V \subset (\mathbb{R}_{>0})^n$ be a set definable in a regular polynomially bounded structure, and let $\pi : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be the projection on the first m coordinates (with $m < n$). Then we have*

$$\mathcal{A}_0(\pi(V)) = \pi(\mathcal{A}_0(V))$$

Proof: It is possible to give a direct proof of this proposition by using the previous theorem. Anyway, we prefer to postpone the proof after corollary 69, where it becomes straightforward. □

Now we suppose that $\mathfrak{R}$ is a regular polynomially bounded structure, or, equivalently, that $\mathfrak{R}$ has field of exponents $\mathbb{R}$. Let $x \in \mathcal{A}_0(V)$. We want to describe a neighborhood of x in $\mathcal{A}_0(V)$. To do this, we choose a map $B \in GL_n(\mathbb{R})$ such that $B(x) = (0, \dots, 0, -1)$. Now we only need to describe a neighborhood of $(0, \dots, 0, -1)$ in $\mathcal{A}_0(\overline{B}(V))$. As logarithmic limit sets are cones, we only need to describe a neighborhood of 0 in

$$H = \{(x_1, \dots, x_{n-1}) \in \mathbb{R}^{n-1} \mid (x_1, \dots, x_{n-1}, -1) \in \mathcal{A}_0(\overline{B}(V))\}$$

For every $t \in (0, 1)$, we define the set:

$$W_t = \{(x_1, \dots, x_{n-1}) \in (\mathbb{R}_{>0})^{n-1} \mid (x_1, \dots, x_{n-1}, t) \in \overline{B}(V)\}$$

Then we define

$$W = \left(\lim_{t \to 0} W_t\right) \cap (\mathbb{R}_{>0})^{n-1}$$

as a limit of a one-parameter family of subsets of $\mathbb{R}^{n-1}$. The set W is a definable subset of $(\mathbb{R}_{>0})^{n-1}$. Its logarithmic limit set is denoted, as usual, by $\mathcal{A}_0(W) \subset \mathbb{R}^{n-1}$. By previous theorem, as $(0,\dots,0,-1) \in \mathcal{A}_0(\overline{B}(V))$, W is not empty, hence $0 \in \mathcal{A}_0(W)$. We want to prove that there exists a neighborhood U of 0 in $\mathbb{R}^{n-1}$ such that $\mathcal{A}_0(W) \cap U = H \cap U$. To do this we will prove that $\mathcal{A}_0(W) \cap H$ is a neighborhood of 0 both in $\mathcal{A}_0(W)$ and in H.

A **flag** in $\mathbb{R}^n$ is a sequence $(V_0, V_1, \dots, V_h)$, $h \leq n$, of subspaces of $\mathbb{R}^n$ such that $V_0 \subset V_1 \subset \cdots \subset V_h \subset \mathbb{R}^n$ and $\dim V_i = i$. We say that a sequence $x(k)$ in $\mathbb{R}^n$ converges to the point y **along** the flag $(V_1, V_2, \dots, V_h)$ if

1. $x(k) \to y$.
2. $\forall k : x(k) - y \in V_h \setminus V_{h-1}$.
3. $\forall i \in \{0, \dots, h-2\}$, the sequence $\pi_i(x(k))$ converges to the point $\pi_i(V_{i+1})$, where $\pi_i : V_h \setminus V_i \longrightarrow \mathbb{P}(V_h/V_i)$ is the canonical projection.

Lemma 49. *For all sequences $x(k)$ in $\mathbb{R}^n$ converging to a point y, there exists a flag $(V_0, \dots, V_h)$ and a subsequence of $x(k)$ converging to y along $(V_0, \dots, V_h)$.*

Proof: It follows from the compactness of $\mathbb{P}(V_h/V_i)$. □

Lemma 50. *Let $x(k) \subset H$ be a sequence converging to 0. Then at least one of its points is in $\mathcal{A}_0(W) \cap H$.*

Proof : We can extract a subsequence, again denoted by $x(k)$, converging to zero along a flag $(V_0, V_1, \dots, V_h)$ in $\mathbb{R}^{n-1}$. Up to a linear change of coordinates, we can suppose that this flag is given by $(\{0\}, \mathrm{Span}(e_{n-1}), \mathrm{Span}(e_{n-2}, e_{n-1}), \dots, \mathrm{Span}(e_{n-h}, \dots, e_{n-1}))$. Hence for $i \in \{1, \dots, n-h-1\}$ we have $x_i(k) = 0$. Again by extracting a subsequence and by a change of coordinates with maps of the form

$$B_i(x_1, \dots, x_i, \dots, x_n) = (x_1, \dots, -x_i, \dots, x_n)$$

with $i \in \{n-h, \dots, n-1\}$, we can suppose that for all such i, $x_i(k) < 0$.

By proposition 36, as $H \subset \mathcal{A}_0(B(V))$, for every point $x(k)$ there exists a sequence $y(k,l)$ in $B(V)$ and a sequence $t(k,l)$ in $(0,1)$ such that $t(k,l) \to 0$ and $\mathrm{Log}_{\left(\frac{1}{t(k,l)}\right)}(y(k,l)) \to x(k)$. By theorem 47 we can choose $y(k,l)$ such that $y(k,l) \to a(k)$, with $a_i(k) > 0$ for $i \in \{1, \dots, n-h-1\}$, and $a_i(k) = 0$ for $i \in \{n-h, \dots, n\}$. Again up to a change of coordinates with maps of the form

$$B_i(x_1, \dots, x_i, \dots, x_n) = (x_1, \dots, -x_i, \dots, x_n)$$

with $i \in \{1, \dots, n-h-1\}$, we can suppose that the sequence $a(k)$ is bounded and that, up to subsequences, it converges to a point a, with $a_i = 0$ for $i \in \{n-h, \dots, n\}$.

Let $\pi : \mathbb{R}^n \longrightarrow \mathbb{R}^{n-h-1}$ be the projection on the first $n-h-1$ coordinates. Then $\pi(a(k)) \subset (\mathbb{R}_{>0})^{n-h-1}$. By lemma 39 we can suppose that $\pi(a(k))$ is in standard position, i.e. $a_1, \ldots, a_g > 0, a_{g+1} = \cdots = a_n = 0$ and:

$$\forall N \in \mathbb{N} : \exists k_0 : \forall k > k_0 : \forall i \in \{g+1, \ldots, n-h-2\} : a_{i+1}(k) < (a_i(k))^N$$

From the sequences $y(k,l)$, we extract a diagonal subsequence $z(k)$ in the following way. For every k, the sequence $y(k,l)$ converges to $a(k) = (a_1(k), \ldots, a_{n-h-1}(k), 0, \ldots, 0)$. As $\mathrm{Log}_{\frac{1}{t(k,l)}}(y(k,l)) \to x(k) = (x_1(k), \ldots, x_{n-h-1}(k), 0, \ldots, 0, -1)$, for all $i \in \{n-h, \ldots, n\}$ we have

$$\frac{\log(y_1(k,l))}{\log(y_i(k,l))} \longrightarrow \frac{-1}{x_i(k)}$$

We can choose an l_0 such that:

1. $\forall i \in \{1, \ldots, h\} : \left| \frac{\log(y_1(k,l_0))}{\log(y_i(k,l_0))} - \frac{-1}{x_i(k)} \right| < \frac{1}{k}$

2. $|y(k,l_0) - a(k)| < \frac{1}{k}$

We define $z(k) = y(k,l_0)$. Now $z(k) \to a = (a_1, \ldots, a_g, 0, \ldots, 0)$ and, as $x(k) \to (0, \ldots, 0, -1)$ along the flag $(V_0, \ldots, V_h)$, we have:

$$\forall N \in \mathbb{N} : \exists k_0 : \forall k > k_0 : \forall i \in \{g+1, \ldots, n-1\} : x_{i+1}(k) < (x_i(k))^N$$

Let r be smaller than any of the numbers $a_1, \ldots, a_g$. Consider the parallelepiped

$$c_r = \{(z_1, \ldots, z_n) \in \mathbb{R}^n \mid |z_1 - a_1| < \tfrac{1}{2}r, \ldots, |z_g - a_g| < \tfrac{1}{2}r\}$$

Let $\pi : \mathbb{R}^n \longrightarrow \mathbb{R}^{n-g}$ be the projection on the last $n-g$ coordinates. The set $\pi(B(V) \cap c_r)$ is definable in $\mathbb{R}^{n-g}$, and the sequence $\pi(z(k))$ satisfies the hypotheses of corollary 44, hence $\pi(B(V) \cap c_r)$ contains a basic exponential cone, hence $\pi(W \cap c_r)$ also contains one. This means that $\mathcal{A}_0(\pi(W) \cap c_r)$ contains a basic cone. Hence also $\mathcal{A}_0(W \cap c_r)$ contains this cone, and also $\mathcal{A}_0(W)$. At least one of the points $x(k)$ is in this cone. □

Lemma 51. *Let $x \in \mathcal{A}_0(W)$. Then the number*

$$r(x) = \sup\{r \mid \forall 0 \leq \lambda \leq r : \lambda x \in H\}$$

is strictly positive.

Proof : Let $x \in \mathcal{A}_0(W)$. By a linear change of coordinates, we can suppose that $x = (0, \ldots, 0, -1, -1)$. By theorem 47 there is a sequence $x(k)$ in W converging to the point $(a_1, \ldots, a_{n-2}, 0)$, with $a_1, \ldots, a_{n-2} > 0$. As W is the limit of the family W_t, for every k there is a sequence $y(k,l)$ in $B(V)$

converging to $(x(k), 0)$. We can construct a diagonal sequence $z(k)$ in the following way: for every k we can choose an l_0 such that

$$|y(k, l_0) - (x(k), 0)| < (x_{n-1}(k))^k$$

The sequence $z(k)$ converges to $(a_1, \dots, a_{n-2}, 0, 0)$. Let r be smaller that any of the $a_1, \dots, a_{n-2}$. Consider the parallelepiped

$$c_r = \{(z_1, \dots, z_n) \in \mathbb{R}^n \mid |z_1 - a_1| < \tfrac{1}{2}r, \dots, |z_{n-2} - a_{n-2}| < \tfrac{1}{2}r\}$$

Let $\pi : \mathbb{R}^n \longrightarrow \mathbb{R}^2$ be the projection on the last 2 coordinates. The set $\pi(B(V) \cap c_r)$ is definable in $\mathbb{R}^2$, and the sequence $\pi(z(k))$ satisfies the hypotheses of corollary 45, hence $\pi(B(V) \cap c_r)$ contains a basic exponential cone. This means that there exists a number $r' > 0$ such that

$$\{(0, \dots, 0, z, -1) \mid -r' \leq z \leq 0\} \subset H$$

□

Theorem 52. *Let $V \subset (\mathbb{R}_{>0})^n$ be a set definable in a regular polynomially bounded structure. Let $x \in \mathcal{A}_0(V)$ and choose a map $B \in GL_n(\mathbb{R})$ such that $B(x) = (0, \dots, 0, -1)$. We recall that*

$$H = \{(x_1, \dots, x_{n-1}) \in \mathbb{R}^{n-1} \mid (x_1, \dots, x_{n-1}, -1) \in \mathcal{A}_0(\overline{B}(V))\}$$

$$W_t = \{(x_1, \dots, x_{n-1}) \in (\mathbb{R}_{>0})^{n-1} \mid (x_1, \dots, x_{n-1}, t) \in \overline{B}(V)\}$$

$$W = \left(\lim_{t \to 0} W_t\right) \cap (\mathbb{R}_{>0})^{n-1}$$

Then there exists a neighborhood U of 0 in $\mathbb{R}^{n-1}$ such that $\mathcal{A}_0(W) \cap U = H \cap U$.

Proof : We will prove that $\mathcal{A}_0(W) \cap H$ is a neighborhood of 0 both in $\mathcal{A}_0(W)$ and in H. Previous lemma implies that if $x(k)$ is a sequence in H converging to 0, then at least one of its points is in $\mathcal{A}_0(W)$, hence $\mathcal{A}_0(W) \cap H$ is a neighborhood of 0 in H.

To prove that $\mathcal{A}_0(W) \cap H$ is also a neighborhood of 0 in $\mathcal{A}_0(W)$, we only need to prove that if r is the function defined in lemma 51, there exists an $\varepsilon > 0$ such that

$$\forall x \in \mathcal{A}_0(W) \cap S^{n-2} : r(x) > \varepsilon$$

But this is true, because we already know that $\mathcal{A}_0(W) \cap H$ is a neighborhood of 0 in H. □

Theorem 53. *Let $V \subset (\mathbb{R}_{>0})^n$ be a set definable in a regular polynomially bounded structure. The logarithmic limit set $\mathcal{A}_0(V)$ is a polyhedral complex. Moreover, if $\dim V = m$, then $\dim \mathcal{A}_0(V) \leq m$.*

Proof : By induction on n. For $n = 1$ the statement is trivial, as a cone in $\mathbb{R}$ is a polyhedral set, and every zero dimensional definable set is compact, hence its logarithmic limit set is a point. Suppose the statement true for $n-1$. For every $x \in \mathcal{A}_0(V)$ there is a linear map B sending x to $(0, \dots, 0, -1)$. The statement in [DM96, 4.7] implies that the definable set $W \subset (\mathbb{R}_{>0})^n$ has dimension less than or equal to $m-1$, hence $\mathcal{A}_0(W)$ is a polyhedral set of dimension less than or equal to $m-1$ (by inductive hypothesis). By previous theorem a neighborhood of the ray $\{\lambda x \mid \lambda \geq 0\}$ in $\mathcal{A}_0(V)$ is the cone over a neighborhood of 0 in $\mathcal{A}_0(W)$, hence it is a polyhedral complex of dimension less than or equal to m. By compactness of the sphere S^{n-1}, $\mathcal{A}_0(V)$ can be covered by a finite number of such neighborhoods, hence it is a polyhedral complex of dimension less than or equal to m. □

Note that the statement about the dimension can be false for a general set. See figure 3.4 for an example.

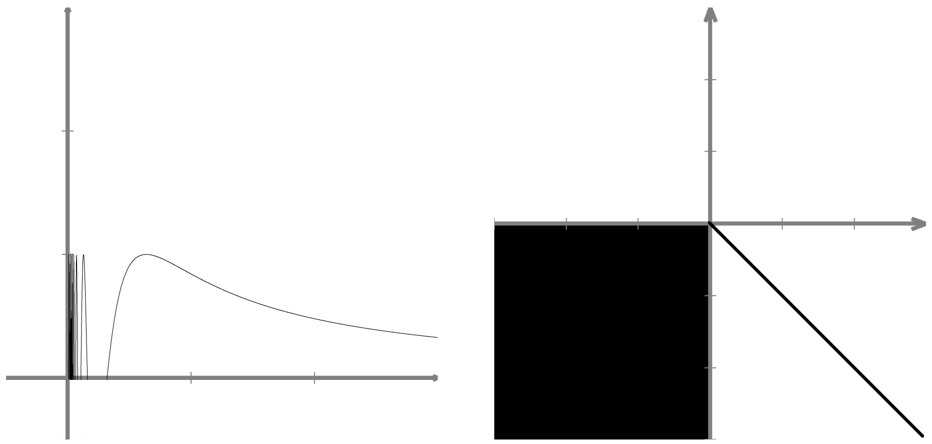

Figure 3.4: $V = \{(x, y) \in (\mathbb{R}_{>0})^2 \mid y = \sin \frac{1}{x}\}$ (left picture), then $\mathcal{A}_0(V) = \{(x, y) \in \mathbb{R}^2 \mid y \leq 0, x \leq 0 \text{ or } x \geq 0, y = -x\}$ (right picture).

Moreover, it is not possible to give more than an inequality, as for every $s \leq m$ it is always possible to find a semi-algebraic set $V \subset (\mathbb{R}_{>0})^m$ such that $\dim V = m$ and $\dim \mathcal{A}_0(V) = s$. For example take the parallelepiped $V = [1, 2]^{m-s} \times (\mathbb{R}_{>0})^s \subset (\mathbb{R}_{>0})^m$, with $\mathcal{A}_0(V) = \{0\}^{m-s} \times (\mathbb{R}_{>0})^s$. It is also possible to find counterexamples of these kind where V is the intersection of $(\mathbb{R}_{>0})^{m+1}$ with an algebraic hypersurface. For example let $S^{m-s} \subset (\mathbb{R}_{>0})^{m-s+1}$ be the sphere with center $(2, \dots, 2)$ and radius 1, then $V = S^{m-s} \times (\mathbb{R}_{>0})^s \subset (\mathbb{R}_{>0})^{m+1}$ has dimension m, but $\mathcal{A}_0(V) = \{0\}^{m-s+1} \times (\mathbb{R}_{>0})^s$ has dimension s.

It is also possible to find a semi-algebraic set V that is the intersection of $(\mathbb{R}_{>0})^n$ with an irreducible pure-dimensional smooth hypersurface, and such that its logarithmic limit set $\mathcal{A}_0(V)$ is not pure-dimensional, see for example figure 3.5. Note that the product $V \times S^h$, with S^h the sphere with center $(2, \dots, 2)$ and radius 1 as above, is again the intersection of $(\mathbb{R}_{>0})^{n+h+1}$ with an irreducible pure-dimensional smooth variety, and its logarithmic limit set is lower dimensional and not pure-dimensional.

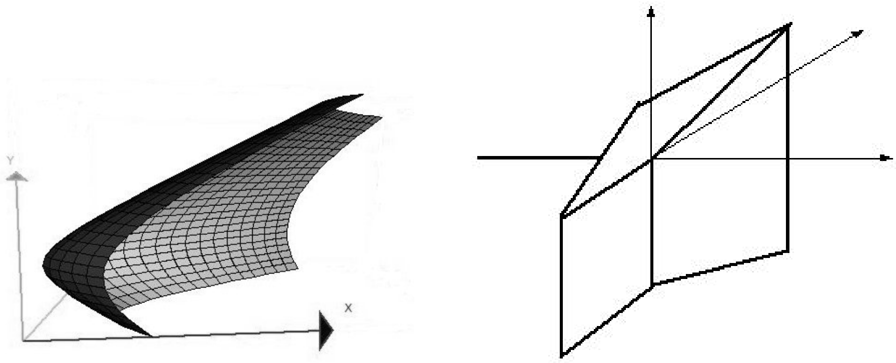

Figure 3.5: $V = \{(x, y, z) \in (\mathbb{R}_{>0})^3 \mid (x+1)^2 = 1 + (y-1)^2 + (z-1)^2 = 0\}$ (left picture), then $\mathcal{A}_0(V)$ has an isolated ray along the direction $(-1, 0, 0)$ and a bi-dimensional part in the half-space $x \geq 0$ (right picture).

3.3 Tropical description

3.3.1 Maslov dequantization

Every real number $t \in (0, 1)$ defines an analytic function:

$$\mathbb{R}_{>0} \ni z \longrightarrow \log_{\left(\frac{1}{t}\right)} z = \left(\frac{-1}{\log t}\right) \log z \in \mathbb{R}$$

This function is bijective, with inverse $x \longrightarrow t^{-x}$, and it preserves the order $\leq$. The operations ('+' and '$\cdot$') are transformed via conjugation in the following way:

$$x \oplus_t y = \log_{\left(\frac{1}{t}\right)}(t^{-x} + t^{-y})$$

$$x \odot_t y = \log_{\left(\frac{1}{t}\right)}(t^{-x} \cdot t^{-y}) = x + y$$

Hence every t induces an $\mathcal{OS}$-structure on $\mathbb{R}$:

$$\mathbb{R}^t = (\mathbb{R}, \{\leq\}, \{\oplus_t, \odot_t\}, \emptyset)$$

This structure is isomorphic to $\mathbb{R}_{>0}$, hence it is an ordered semifield, i.e. it respects all properties of ordered fields except for the addiction that is not invertible.

In the limit for t tending to zero we have:

$$\lim_{t \to 0^+} x \oplus_t y = \max(x, y)$$

The limit $\mathcal{OS}$-structure is

$$\mathbb{R}^{trop} = (\mathbb{R}, \{\leq\}, \{\max, +\}, \emptyset)$$

the **tropical semifield**. This structure is again an ordered semifield, but is not isomorphic to $\mathbb{R}_{>0}$ any more, as the addition is idempotent. We will denote its operations by $\oplus = \max$ and $\odot = +$. Note that if $y \leq x$ we have $x \oplus y = x$ and

$$x \leq \log_{\left(\frac{1}{t}\right)}(t^{-x} + t^{-y}) \leq \log_{\left(\frac{1}{t}\right)}(2t^{-x}) = \log_{\left(\frac{1}{t}\right)} 2 + x$$

This implies the following inequalities:

$$x \oplus y \leq x \oplus_t y \leq x \oplus y + \log_{\left(\frac{1}{t}\right)} 2$$

$$x_1 \oplus \cdots \oplus x_n \leq x_1 \oplus_t \cdots \oplus_t x_n \leq x_1 \oplus \cdots \oplus x_n + \log_{\left(\frac{1}{t}\right)} n$$

In other words the convergence of the family $\mathbb{R}^t$ to the structure $\mathbb{R}^{trop}$ is uniform. This construction is usually called **Maslov dequantization**.

Note that if $\alpha \in \mathbb{R}_{>0}$, the function

$$\mathbb{R}_{>0} \ni x \longrightarrow x^\alpha \in \mathbb{R}_{>0}$$

is transformed, via conjugation with the map $\log_{\left(\frac{1}{t}\right)}$, in the map:

$$\mathbb{R} \ni x \longrightarrow \log_{\left(\frac{1}{t}\right)} \left((t^{-x})^\alpha\right) = \alpha x$$

As this map does not depend on t, it induces also a map in the limit structure $\mathbb{R}^{trop}$. With these maps, the structures $\mathbb{R}^t$ and $\mathbb{R}^{trop}$ are turned in $\mathcal{OS}^{\mathbb{R}}$-structures.

The family of maps Log_t, which we used to construct the logarithmic limit sets, is the Maslov dequantization applied coordinate-wise to $(\mathbb{R}_{>0})^n$.

3.3.2 Dequantization of formulae

An $L_{\mathcal{OS}^{\mathbb{R}}}$-term can be defined inductively in this way (see [EFT84, Chap. II, def. 3.1]):

1. Every variable is a term.
2. If u, v are terms, then $(u + v)$ and $(u \cdot v)$ are terms.
3. If u is a term and $\alpha \in \mathbb{R}$, then (u^α) is a term.

A term $u(x_1, \dots, x_n, y_1, \dots y_m)$ with variables $x_1, \dots, x_n, y_1, \dots, y_m$ and constants $a_1, \dots, a_m \in \mathbb{R}_{>0}$ defines a function:

$$U : (\mathbb{R}_{>0})^n \ni (x_1, \dots, x_n) \longrightarrow u(x_1, \dots, x_n, a_1, \dots, a_m) \in \mathbb{R}_{>0}$$

For every t, this function defines, by conjugation with the map $\log_{\left(\frac{1}{t}\right)}$, a function on $\mathbb{R}^n$ corresponding to the term u where the operations are interpreted with the operations of $\mathbb{R}^t$, and every constant a_i is interpreted as $\log_{\left(\frac{1}{t}\right)}(a_i)$:

$$U_t = \log_{\left(\frac{1}{t}\right)} \circ U \circ \left(\mathrm{Log}_{\left(\frac{1}{t}\right)}\right)^{-1} : \mathbb{R}^n \longrightarrow \mathbb{R}$$

Lemma 54. *Let $U_0 : \mathbb{R}^n \longrightarrow \mathbb{R}$ be the function defined by the term u where the operations are interpreted with the operations of $\mathbb{R}^{trop}$, and every constant a_i is interpreted as 0. Then*

$$\forall x \in \mathbb{R}^n : U_0(x) \leq U_t(x) \leq U_0(x) + \log_{\left(\frac{1}{t}\right)} C$$

where C is a constant depending only on the term u and the coefficients a_i. In particular the family of functions U_t uniformly converges to the function U_0.

Proof : By induction on the complexity of the term. If $u = x_1$, then $U_0 = U_t$ and $C = 1$. If $u = y_1$ then $U_t = \log_{\left(\frac{1}{t}\right)} a_1$ and $U_0 = 0$, hence $C = a_1$.

If $u = v^\alpha$, where $\alpha \in \mathbb{R}$, then

$$V_0 \leq V_t \leq V_0 + \log_{\left(\frac{1}{t}\right)} C$$

As $U_0 = \alpha V_0$ and $U_t = \alpha V_t$ we have

$$U_0 \leq U_t \leq U_0 + \log_{\left(\frac{1}{t}\right)} C^\alpha$$

If $u = v \cdot w$

$$V_0 \leq V_t \leq V_0 + \log_{\left(\frac{1}{t}\right)} C$$

$$W_0 \leq W_t \leq W_0 + \log_{\left(\frac{1}{t}\right)} D$$

As $U_0 = V_0 + W_0$ and $U_t = V_t + W_t$, we have

$$U_0 \leq U_t \leq U_0 + \log_{\left(\frac{1}{t}\right)} CD$$

If $u = v + w$

$$V_0 \leq V_t \leq V_0 + \log_{\left(\frac{1}{t}\right)} C$$

$$W_0 \leq W_t \leq W_0 + \log_{\left(\frac{1}{t}\right)} D$$

As $U_0 = \max(V_0, W_0)$ and

$$\max(V_0, W_0) \leq U_t \leq \max(V_t, W_t) + \log_{\left(\frac{1}{t}\right)} 2 \leq$$

$$\leq \max(V_0, W_0) + \log_{\left(\frac{1}{t}\right)} \max(C, D) + \log_{\left(\frac{1}{t}\right)} 2$$

we have

$$U_0 \leq U_t \leq U_0 + \log_{\left(\frac{1}{t}\right)} 2 \max(C, D)$$

□

If $\phi(x_1, \dots, x_n, y_1, \dots, y_m)$ is an $L_{\mathcal{OS}^{\mathbb{R}}}$-formula and $a_1, \dots, a_m$ are constants, they define the set:

$$V = \{(x_1, \dots, x_n) \in (\mathbb{R}_{>0})^n \mid \phi(x_1, \dots, x_n, a_1, \dots, a_m)\}$$

We will denote by ϕ_t the formula ϕ where the operations are interpreted in the structure $\mathbb{R}^t$, and ϕ_0 the formula ϕ where the operations are interpreted in the structure $\mathbb{R}^{trop}$. Hence

$$\mathcal{A}_t(V) = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid \phi_t(x_1, \dots, x_n, \log_{\left(\frac{1}{t}\right)} a_1, \dots, \log_{\left(\frac{1}{t}\right)} a_m)\}$$

Because $\log_{\left(\frac{1}{t}\right)}$ is a semifield isomorphism hence the amoeba $\mathcal{A}_t(V)$ is described by the same formula.

Anyway it is not always true that

$$\mathcal{A}_0(V) = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid \phi_t(x_1, \dots, x_n, 0, \dots, 0\}$$

For example if $\phi(x_1) = \neg(x \leq 1)$, then $\phi_0 = \neg(x \leq 0)$, but the logarithmic limit set of $\{x > 1\}$ is not $\{x > 0\}$, but $\{x \geq 0\}$.

3.3.3 Dequantization of sets

A **positive formula** is a formula written without the symbols $\neg, \Rightarrow, \Leftrightarrow$. These formulae contains only the connectives $\vee$ and $\wedge$ and the quantifiers $\forall$, $\exists$. Consider the standard $\mathcal{OS}^{\mathbb{R}}$-structure on $\mathbb{R}_{>0}$, or one of the $\mathcal{OS}^{\mathbb{R}}$-structures $\mathbb{R}^t$ or $\mathbb{R}^{trop}$ on $\mathbb{R}$. Every subset of $(\mathbb{R}_{>0})^n$ or $\mathbb{R}^n$ that is defined by a quantifier-free positive $L_{\mathcal{OS}^{\mathbb{R}}}$-formula in one of these structures is closed, as the set of symbols $\mathcal{OS}^{\mathbb{R}}$ has only the relations $=$ and $\leq$, that are closed, and the functions $+$, $\cdot$, x^α, that are continuous.

Proposition 55. *Let $\phi(x_1, \dots, x_n, y_1, \dots, y_m)$ be a positive $L_{\mathcal{OS}^{\mathbb{R}}}$-formula, let $a_1, \dots, a_m \in \mathbb{R}_{>0}$ be parameters and $V \subset (\mathbb{R}_{>0})^n$ be a set such that*

$$V \subset \{(x_1, \dots, x_n) \in (\mathbb{R}_{>0})^n \mid \phi(x_1, \dots, x_n, a_1, \dots, a_m)\}$$

Then

$$\mathcal{A}_0(V) \subset \{x \in \mathbb{R}^n \mid \phi_0(x_1, \ldots, x_n, 0, \ldots, 0)\}$$

Proof : By induction on the complexity of the formula. If ϕ is atomic, then it has the one of the forms

$$u(x_1, \ldots, x_n, y_1, \ldots, y_m) = v(x_1, \ldots, x_n, y_1, \ldots, y_m)$$

$$u(x_1, \ldots, x_n, y_1, \ldots, y_m) \leq v(x_1, \ldots, x_n, y_1, \ldots, y_m)$$

We have

$$\mathcal{A}_t(V) \subset \{x \in \mathbb{R}^n \mid \phi_t(x_1, \ldots, x_n, \log_{\frac{1}{t}}(a_1), \ldots, \log_{\frac{1}{t}}(a_m)\}$$

We may put all the equations together, one for every t, thus finding a description for the deformation

$$\mathcal{D} = \{(x, t) \in \mathbb{R}^n \times (0, \varepsilon) \mid x \in \mathcal{A}_t(V)\}$$

$$\mathcal{D} = \{(x_1, \ldots, x_n, t) \in \mathbb{R}^n \times (0, 1) \mid \phi_t(x_1, \ldots, x_n, \log_{\frac{1}{t}}(a_1), \ldots, \log_{\frac{1}{t}}(a_m)\}$$

If we consider U_t and V_t as functions on $\mathbb{R}^n \times (0, 1)$, they can be extended continuously to $\mathbb{R}^n \times [0, 1)$ defining the extensions on $\mathbb{R}^n \times \{0\}$ by U_0, V_0. Hence we get following inclusion for the logarithmic limit set:

$$\mathcal{A}_0(V) \subset \{x \in \mathbb{R}^n \mid \phi_0(x_1, \ldots, x_n, 0, \ldots, 0)\}$$

If $\phi = \psi_1 \vee \psi_2$, then $V \subset V_1 \cup V_2$, where V_i is defined by ψ_i. The statement follows from the fact that the logarithmic limit set of a union is the union of the logarithmic limit sets (see proposition 36).

If $\phi = \psi_1 \wedge \psi_2$, then $V \subset V_1 \cap V_2$, where V_i is defined by ψ_i. The statement follows from the fact that the logarithmic limit set of an intersection is contained in the intersection of the logarithmic limit sets (see proposition 36).

If $\phi = \exists x_{n+1} : \psi$, then V is contained in the projection of W, where W is the set defined by ψ. The statement follows from the fact that the logarithmic limit set of the projection is the projection of the logarithmic limit sets (see proposition 48).

If $\phi = \forall x_{x+1} : \psi$, then we denote by W the set defined by ψ. If $(0, \ldots, 0, -1) \in \mathcal{A}_0(V)$, there is a sequence $x(k)$ in V converging to a point $(b_1, \ldots, b_{n-1}, 0)$ with $b_i \neq 0$. Then W contains a sequence of lines $\{(x(k), y) \mid y \in \mathbb{R}_{>0}\}$, hence $\mathcal{A}_0(W)$ contains the line $\{(0, \ldots, 0, y) \mid y \in \mathbb{R}\}$. As $\mathcal{A}_0(W) \subset \{(x_1, \ldots, x_{n+1}) \mid \psi_0(x_1, \ldots, x_{n+1})\}$, then $\mathcal{A}_0(V) \subset \{(x_1, \ldots, x_n) \mid \forall x_{n+1} : \psi_0(x_1, \ldots, x_{n+1})\}$. $\square$

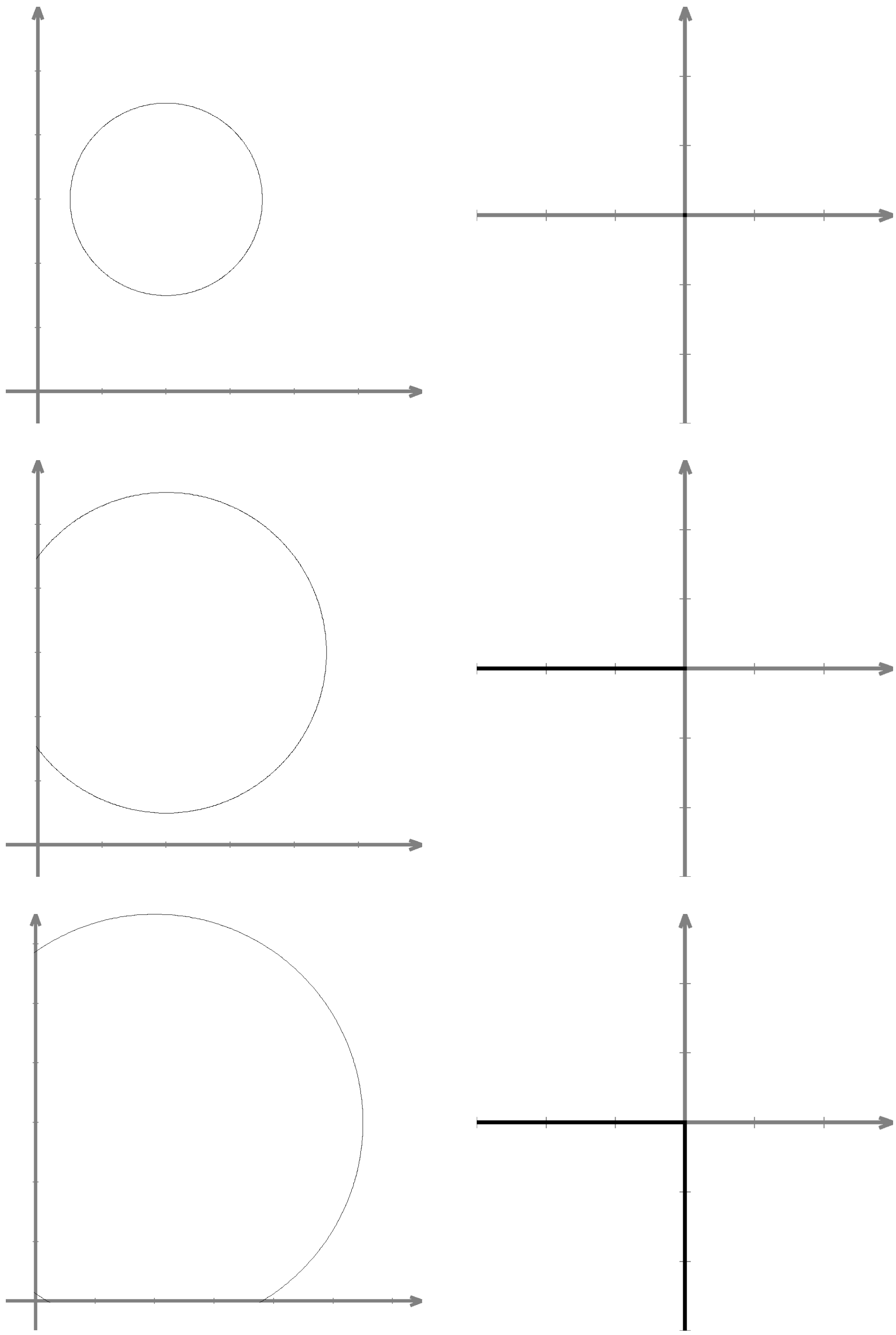

Figure 3.6: $V_r = \{(x, y) \in (\mathbb{R}_{>0})^2 \mid x^2+y^2+13 = r^2+4x+6y\}$ for $r = \frac{3}{2}, \frac{5}{2}, \frac{7}{2}$ respectively (on the left), and their logarithmic limit sets (on the right).

Anyway there are examples where

$$V = \{x \in (\mathbb{R}_{>0})^n \mid \phi(x_1, \dots, x_n, a_1, \dots, a_m)\}$$

with ϕ a positive $\mathcal{OS}^{\mathbb{R}}$-formula, and

$$\mathcal{A}_0(V) \subsetneq \{x \in \mathbb{R}^n \mid \phi_0(x_1, \dots, x_n, 0, \dots, 0)\}$$

For example consider the following atomic formula $\phi(x_1, x_2, y_1, y_2, y_3)$:

$$x_1^2 + x_2^2 + y_1 = y_2 x_1 + y_3 x_2$$

with constants $a_1 = 13 - r^2, a_2 = 4, a_3 = 6$, with $r^2 < 13$. This derives from the equation of a circle with center in $(2, 3)$ and radius r:

$$(x_1 - 2)^2 + (x_2 - 3)^2 = r^2$$

The dequantized formula $\phi_0(x_1, x_2, 0, 0, 0, 0)$ does not depend on the value of r:

$$\max(2x_1, 2x_2, 0) = \max(x_1, x_2)$$

Now if

$$V_r = \{(x_1, x_2) \in (\mathbb{R}_{>0})^n \mid x_1^2 + x_2^2 + 13 - r^2 = 4x_1 + 6x_2\}$$

the logarithmic limit sets of $V_{\frac{3}{2}}$, $V_{\frac{5}{2}}$, $V_{\frac{7}{2}}$ are different (see figure 3.6), we have that

$$\mathcal{A}_0(V_{\frac{7}{2}}) = \{(x_1, x_2) \in \mathbb{R}^2 \mid \max(2x_1, 2x_2, 0) = \max(0, x_1, x_2)\}$$

but for $\mathcal{A}_0(V_{\frac{3}{2}})$ and $\mathcal{A}_0(V_{\frac{5}{2}})$ we have a strict inclusion.

Even if $\phi(x_1, x_2, \frac{27}{4}, 4, 6)$ is a definition of $V_{\frac{5}{2}}$, $\phi_0(x_1, x_2, 0, 0, 0)$ is not a definition of $\mathcal{A}_0(V_{\frac{5}{2}})$. Anyway we can find another formula with this property. The equation of $V_{\frac{5}{2}}$ we used in the formula ϕ is:

$$x_1^2 - 4x_1 + x_2^2 - 6x_2 + \frac{27}{4} = 0$$

Another equation is

$$(4x_1^3 + 16x_1^2 + 37x_1 + 40)(x_1^2 - 4x_1 + x_2^2 - 6x_2 + \frac{27}{4}) = 0$$

Or, more explicitly:

$$4x_1^5 + \frac{359}{4} x_1 + 270 + 4x_1^3 x_2^2 + 16x_1^2 x_2^2 + 37x_1 x_2^2 + 40x_2^2 =$$

$$= 24x_1^3 x_2 + 96x_1^2 x_2 + 222x_1 x_2 + 240x_2$$

The dequantized version of this formula is

$$\max(5x_1, x_1, 0, 3x_1 + 2x_2, 2x_1 + 2x_2, x_1 + 2x_2, 2x_2) =$$

$$= \max(3x_1 + x_2, 2x_1 + x_2, x_1 + x_2, x_2)$$

That is equivalent to:

$$\max(5x_1, 0, 2x_2, 3x_1 + 2x_2) = x_2 + \max(3x_1, 0)$$

And this formula is an exact description of $\mathcal{A}_0(V_{\frac{5}{2}})$.

There are examples of real algebraic sets V where it is not possible to find an algebraic formula ϕ defining V such that ϕ_0 is a definition of $\mathcal{A}_0(V)$. Here by an algebraic formula we mean an atomic formula with the relation $=$, in other words an equation between two positive polynomials.

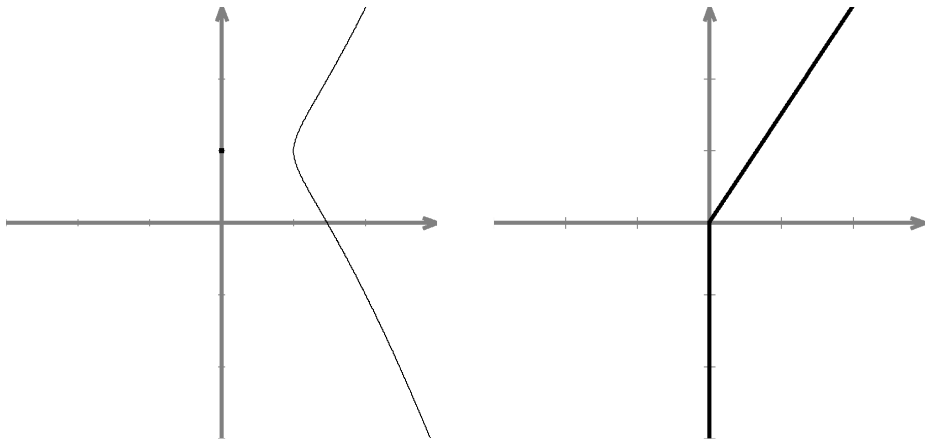

Figure 3.7: $V = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 + 1 = 2y + x^3\}$ with an isolated point in $(0, 1)$ (left picture), then $\mathcal{A}_0(V \cap (\mathbb{R}_{>0})^2) = \{(x, y) \in \mathbb{R}^2 \mid x = 0, y \leq 0 \text{ or } x \geq 0, 2y = 3x\}$ (right picture).

Consider for example the cubic

$$V = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1^2 + x_2^2 + 1 = 2x_2 + x_1^3\}$$

as in figure 3.7. This cubic has an isolated point in $(0, 1)$. This point is outside the positive orthant $(\mathbb{R}_{>0})^2$, hence it does not influence the logarithmic limit set of $V' = V \cap (\mathbb{R}_{>0})^2$, but the set defined by

$$\{(x_1, x_2) \in \mathbb{R}^2 \mid \max(2x_1, 2x_2, 0) = \max(x_2, 3x_1)\}$$

contains also the half line $\{x_2 = 0, x_1 \leq 0\}$ that is not in the logarithmic limit set, and the same happens for every polynomial equation defining V.

We need to use the order relation $\leq$ to construct a formula ϕ defining V' such that ϕ_0 is a definition of $\mathcal{A}_0(V')$. For example:

$$V' = \{(x_1, x_2) \in (\mathbb{R}_{>0})^2 \mid x_1^2 + x_2^2 + 1 = 2x_2 + x_1^3 \ \wedge \ x_1 \geq \frac{1}{2}\}$$

$$\mathcal{A}_0(V') = \{(x_1, x_2) \in \mathbb{R}^2 \mid \max(2x_1, 2x_2, 0) = \max(x_2, 3x_1) \ \wedge \ x_1 \geq 0\}$$

As we will see in the next subsection, this property is a general fact.

3.3.4 Exact definition

Let C be an open convex set such that $(0, \dots, 0, -1) \in C$, and the closure $\overline{C}$ is a convex polyhedral cone contained in $\{x \in \mathbb{R}^n \mid x_n < 0\} \cup \{0\}$. The faces $F_1, \dots, F_k$ of $\overline{C}$ are described by equations

$$a_i^1 x_1 + \dots + a_i^{n-1} x_{n-1} + x_n = 0$$

and C is described by

$$C = \{x \in \mathbb{R}^n \mid x_n < 0 \text{ and } \forall i \in \{1, \dots, k\} : a_i^1 x_1 + \dots + a_i^{n-1} x_{n-1} + x_n < 0\}$$

For every $h \in \mathbb{R}_{>0}$, consider the set

$$E_h(C) = \{x \in (\mathbb{R}_{>0})^n \mid x_n < h \text{ and } \forall i \in \{1, \dots, k\} : x_1^{a_i^1} \dots x_{n-1}^{a_i^{n-1}} x_n < h\}$$

Lemma 56. *Let $V \subset (\mathbb{R}_{>0})^n$ be a set such that $\mathcal{A}_0(V) \cap C = \emptyset$. Then for every sufficiently small $h \in \mathbb{R}_{>0}$ we have $V \cap E_h(C) = \emptyset$.*

Proof: Suppose that for all $i \in \mathbb{N}$ there exists $x_i \in V \cap E_{\frac{1}{i}}(C)$. Then from the sequence $(x_i) \subset V$ we can extract a subsequence y_i such that $\mathrm{Log}_e(y_i)$ converges to a point $y \in C$. □

Note that $E_h(C)$ can be described by the following $S_{OS}^{\mathbb{R}}$-formula, with $y = h$

$$\phi^C(x_1, \dots, x_n, y) = \neg(y \leq x_n \vee y \leq x_1^{a_1^1} \dots x_{n-1}^{a_1^{n-1}} x_n \vee \dots \vee y \leq x_1^{a_n^1} \dots x_{n-1}^{a_n^{n-1}} x_n)$$

and C is described by the formula ϕ_0^C with $y = 0$.

Let $C \subset \mathbb{R}^n$ be an open convex set such that the closure $\overline{C}$ is a convex polyhedral cone and $\overline{C} \subset H \cup \{0\}$ where H is an open half-space H.

There exists a linear map B such that $(0, \dots, 0, -1) \in B(C)$, and $B(\overline{C})$ is contained in $\{x \in \mathbb{R}^n \mid x_n < 0\} \cup \{0\}$. We will use the notation

$$E_h(C) = \overline{B}^{-1}(E_h(B(C)))$$

As before there exists a $S_{OS}^{\mathbb{R}}$-formula $\phi^C(x_1, \dots, x_n, y)$ such that

$$E_h(C) = \{x \in (\mathbb{R}_{>0})^n \mid \phi^C(x_1, \dots, x_n, h)\}$$

$$C = \{x \in \mathbb{R}^n \mid \phi_0^C(x_1, \ldots, x_n, 0)\}$$

Let $V \subset (\mathbb{R}_{>0})^n$ be a set definable in an o-minimal, polynomially bounded structure with field of exponents $\mathbb{R}$. Then, by theorem 53, $\mathcal{A}_0(V)$ is a polyhedral complex, hence we can find a finite number of sets $C_1, \ldots, C_k$ such that sets such that

1. $C_1 \cup \cdots \cup C_k$ is the complement of $\mathcal{A}_0(V)$.
2. The closure $\overline{C_i}$ is a convex polyhedral cone.
3. There exists an open half-space H_i such that $\overline{C_i} \subset H_i \cup \{0\}$.

Lemma 57. *Consider the $S_{OS}^{\mathbb{R}}$-formula*

$$\phi(x_1, \ldots, x_n, y) = \neg\phi^{C_1}(x_1, \ldots, x_n, y) \wedge \cdots \wedge \neg\phi^{C_k}(x_1, \ldots, x_n, y))$$

Then

$$\mathcal{A}_0(V) = \{x \in \mathbb{R}^n \mid \phi_0(x_1, \ldots, x_n, 0)\}$$

and for every sufficiently small $h \in \mathbb{R}^n_{>0}$ we have

$$V \subset \{x \in (\mathbb{R}_{>0})^n \mid \phi(x_1, \ldots, x_n, h)\}$$

Proof : The first assertion is trivial, and the second assertion follows from previous lemma. □

Note that the formula ϕ of previous lemma is equivalent to a formula of the form:

$$\psi_1 \wedge \cdots \wedge \psi_k$$

where ψ_i have the form:

$$\psi_i = y \leq x_1^{a_1^1} \ldots x_n^{a_1^n} \vee \cdots \vee y \leq x_1^{a_m^1} \ldots x_n^{a_m^n}$$

These formulae does not contain the $+$ operation, hence when they are interpreted with the dequantizing operations $\oplus_t, \odot_t$ or the tropical operations $\oplus, \odot$ the interpretation does not depend on t, and it is simply:

$$\psi_i = y \leq a_1^1 x_1 + \cdots + a_1^n x_n \vee \cdots \vee y \leq a_m^1 x_1 + \cdots + a_m^n x_n$$

Corollary 58. *Let V be definable in an o-minimal, polynomially bounded structure with field of exponents $\mathbb{R}$. For all $\varepsilon > 0$, for all sufficiently small $t > 0$*

$$\sup_{x \in \mathcal{A}_t(V)} d(x, \mathcal{A}_0(V)) < \varepsilon$$

Proof : Choose h such that $V \subset \{\phi(x_1, \ldots, x_n, h)\}$. Then $\mathcal{A}_t(V) \subset \{\phi_t(x_1, \ldots, x_n, \log_{(\frac{1}{t})} h)\}$. Note that $\{\phi_t(x_1, \ldots, x_n, \log_{(\frac{1}{t})} h)\}$ is a uniformly bounded neighborhood of $\mathcal{A}_0(V)$, with distance depending linearly on y, hence the distance tends to zero when y tends to zero. □

Theorem 59. *Let $\phi(x_1, \ldots, x_n, y_1, \ldots, y_m)$ be a positive $L_{\mathcal{OS}^{\mathbb{R}}}$-formula, let $a_1, \ldots, a_m \in \mathbb{R}_{>0}$ be parameters and denote*

$$V = \{(x_1, \ldots, x_n) \in (\mathbb{R}_{>0})^n \mid \phi(x_1, \ldots, x_n, a_1, \ldots, a_m)\}$$

Then there exists a positive $L_{\mathcal{OS}^{\mathbb{R}}}$-formula $\psi(x_1, \ldots, x_n, y_1, \ldots, y_l)$ and parameters $b_1, \ldots, b_l \in \mathbb{R}_{>0}$ such that:

$$V = \{(x_1, \ldots, x_n) \in (\mathbb{R}_{>0})^n \mid \psi(x_1, \ldots, x_n, b_1, \ldots, b_l)\}$$

$$\mathcal{A}_0(V) = \{x \in \mathbb{R}^n \mid \psi_0(x_1, \ldots, x_n, 0, \ldots, 0)\}$$

Proof : Let $\phi'(x_1, \ldots, x_n, y)$ and h as in lemma 57. Then $\psi = \phi \wedge \phi'$ is the searched formula. □

Corollary 60. *Let $V \subset (\mathbb{R}_{>0})^n$ be a closed semi-algebraic set. Then there exists a positive quantifier-free $L_{\mathcal{OS}^{\mathbb{R}}}$-formula $\phi(x_1, \ldots, x_n, y_1, \ldots, y_m)$ and constants $a_1, \ldots, a_m \in \mathbb{R}_{>0}$ such that*

$$V = \{x \in (\mathbb{R}_{>0})^n \mid \phi(x_1, \ldots, x_n, a_1, \ldots, a_m)\}$$

$$\mathcal{A}_0(V) = \{x \in \mathbb{R}^n \mid \phi_0(x_1, \ldots, x_n, 0, \ldots, 0)\}$$

Proof : By the Finiteness Theorem (see [BCR98, thm. 2.7.2]), every closed semi-algebraic set is defined by a positive quantifier-free $L_{\mathcal{OS}}$-formula. □

Chapter 4

Non archimedean amoebas for real closed fields

In this chapter we continue the study of the logarithmic limit sets of real semi-algebraic sets that was started in the previous chapter. Here we show how the relation between tropical varieties and images of varieties defined over non-archimedean fields, well known for algebraically closed fields, can be extended to the case of real closed fields. We give the notion of non-archimedean amoebas of semi-algebraic sets and sets definable in other o-minimal structures and we study their relations with logarithmic limit sets of definable sets in $\mathbb{R}^n$, and with patchworking families of definable sets. Note that this notion generalizes the notion of non-archimedean amoebas of semi-linear sets that have been used in [DY] to study tropical polytopes.

In section 4.1 we consider a special class of non-archimedean fields: the Hardy fields of regular polynomially bounded structures. These are non-archimedean real closed fields of rank one extending $\mathbb{R}$, with a canonical real valued valuation and residue field $\mathbb{R}$. The elements of these fields are germs of definable functions, hence they have better geometric properties than the fields of formal series usually employed in tropical geometry. The image, under the component-wise valuation map, of definable sets in the Hardy fields are related with the logarithmic limit sets of real definable sets, and with the limit of real patchworking families.

In section 4.2 we compare the construction of this paper with other known constructions. We show that the logarithmic limit sets of complex algebraic sets are only a particular case of the logarithmic limit sets of real semi-algebraic sets, and the same happens for the limit of complex patchworking families. Hence our methods provide an alternative proof (with a topological flavor) for some known results about complex sets. We also compare the logarithmic limit sets of real algebraic sets with the construction of Positive Tropical Varieties (see [SW]). Even if in many examples these two notions coincide, we show some examples where they differ.

4.1 Non-archimedean description

4.1.1 The Hardy field

Let S be a set of symbols expanding $\mathcal{OS}$, and let $\mathfrak{R} = (\mathbb{R}, a)$ be an o-minimal S-structure expanding $\overline{\mathbb{R}}$ (see subsection 3.1.3 for definitions).

The **Hardy field** of $\mathfrak{R}$ can be defined in the following way. If $f, g : \mathbb{R}_{>0} \longrightarrow \mathbb{R}$ are two definable functions, we say that they have the same **germ** near zero, and we write $f \sim g$, if there exists an $\varepsilon > 0$ such that $f_{|(0,\varepsilon)} = g_{|(0,\varepsilon)}$. The Hardy field can be defined as the set of germs of definable functions near zero, or as

$$H(\mathfrak{R}) = \{f : \mathbb{R}_{>0} \longrightarrow \mathbb{R} \mid f \text{ definable } \}/\sim$$

We will denote by $[f]$ the germ of a function f.

Every relation in the structure $\mathfrak{R}$ defines a relation on $H(\mathfrak{R})$: let $R \subset \mathbb{R}^n$ be an n-ary relation in $\mathfrak{R}$, and let $f_1, \ldots, f_n$ be definable functions. The set

$$\{t \in \mathbb{R}_{>0} \mid R(f_1(t), \ldots, f_n(t))\}$$

is definable, hence either it or its complement contains an interval of the form $(0, \delta)$. If this set contains such an interval, we will write $R([f_1], \ldots, [f_n])$. The definition does not depend on the choice of representatives.

Every function in the structure $\mathfrak{R}$ defines a function on $H(\mathfrak{R})$: let $F : \mathbb{R}^n \longrightarrow \mathbb{R}$ be an n-ary function, and let $f_1, \ldots, f_n$ be definable functions. The function

$$\mathbb{R}_{>0} \ni t \longrightarrow F(f_1(t), \ldots, f_n(t)) \in \mathbb{R}$$

is definable, hence it defines an element of the Hardy field denoted by $F([f_1], \ldots, [f_n])$. Again this definition does not depend on the choice of representatives.

Hence the Hardy field $H(\mathfrak{R})$ can be endowed with an S-structure $\overline{H(\mathfrak{R})}$.

Theorem 61. *Given an (L_S)-formula $\phi(x_1, \ldots, x_n)$, and given definable functions $f_1, \ldots, f_n$, we have:*

$$\overline{H(\mathfrak{R})} \models \phi([f_1], \ldots, [f_n]) \Leftrightarrow \exists \varepsilon > 0 : \forall t \in (0, \varepsilon) : \mathfrak{R} \models \phi(f_1(t), \ldots, f_n(t))$$

Proof : By induction on the complexity of the formula. We can suppose that the formulae are written using only the symbols $\wedge, \neg, \exists$. For atomic formulas the statement holds by definition of the relations. Every formula $\phi(x_1, \ldots, x_n)$ has one of the forms:

1. $\phi(x_1, \ldots, x_n) = \psi_1(x_1, \ldots, x_n) \wedge \psi_2(x_1, \ldots, x_n)$
2. $\phi(x_1, \ldots, x_n) = \neg\psi(x_1, \ldots, x_n)$
3. $\phi(x_1, \ldots, x_n) = \exists x : \psi(x, x_1, \ldots, x_n)$

The first two cases are easy. We prove the third one:

If $\overline{H(\mathfrak{R})} \models \exists x : \psi(x, [f_1], \ldots, [f_n])$, then there exists a definable function f such that $\overline{H(\mathfrak{R})} \models \psi([f], [f_1], \ldots, [f_n])$. Then, by inductive hypothesis, $\exists \varepsilon > 0 : \forall t \in (0, \varepsilon) : \mathfrak{R} \models \psi(f(t), f_1(t), \ldots, f_n(t))$. Hence $\exists \varepsilon > 0 : \forall t \in (0, \varepsilon) : \mathfrak{R} \models \exists x : \psi(x, f_1(t), \ldots, f_n(t))$.

To prove the converse, we will use the existence of the **definable choice functions**, i.e. given a definable set $A \subset \mathbb{R}^{n+m}$, if $\pi : \mathbb{R}^{n+m} \longrightarrow \mathbb{R}^n$ is the projection on the first n coordinates, then there is a definable map $c : \pi(A) \longrightarrow \mathbb{R}^m$ such that for all $x \in \pi(A)$, $(x, c(x)) \in A$ (see [Dr, chap. 6]) for details.

If $\exists \varepsilon > 0 : \forall t \in (0, \varepsilon) : \mathfrak{R} \models \exists x : \psi(x, f_1(t), \ldots, f_n(t))$, then the set

$$\{(x, t) \in \mathbb{R} \times (0, \varepsilon) \mid \psi(x, f_1(t), \ldots, f_n(t))\}$$

is definable in $\mathfrak{R}$, and there exists a definable choice function $f : (0, \varepsilon) \longrightarrow \mathbb{R}$ such that $\forall t \in (0, \varepsilon) : \psi(f(t), f_1(t), \ldots, f_n(t))$. Hence $\exists \varepsilon > 0 : \forall t \in (0, \varepsilon) : \mathfrak{R} \models \psi(f(t), f_1(t), \ldots, f_n(t))$, hence, by inductive hypothesis, $\overline{H(\mathfrak{R})} \models \psi([f], [f_1], \ldots, [f_n])$. Hence $\overline{H(\mathfrak{R})} \models \exists x : \psi(x, [f_1], \ldots, [f_n])$. □

For every element $a \in \mathbb{R}$, the constant function with value a defines a germ that is identified with a. This defines an an embedding $\mathbb{R} \longrightarrow H(\mathfrak{R})$.

Corollary 62. *The S-structure $\overline{H(\mathfrak{R})}$ is an* **elementary extension** *of the S-structure $\mathfrak{R}$. In particular the S-structures $\mathfrak{R}$ and $\overline{H(\mathfrak{R})}$ are* **elementarily equivalent**. □

By this theorem it is clear that the operations $+$ and $\cdot$ turn $H(\mathfrak{R})$ in a field, the order $\leq$ turn it in a ordered field, and that this field is real closed. Moreover, the structure $\overline{H(\mathfrak{R})}$ is o-minimal as it is elementarily equivalent to the o-minimal structure $\mathfrak{R}$.

Suppose that S' is an expansion of S, and that $\mathfrak{R}'$ is an S'-structure expanding $\mathfrak{R}$. Then all functions that are definable in $\mathfrak{R}$ are also definable in $\mathfrak{R}'$. This defines an inclusion $H(\mathfrak{R}) \subset H(\mathfrak{R}')$. Note that, by restriction, $\mathfrak{R}'$ has an S-structure induced by his S'-structure.

Corollary 63. *Let $\mathfrak{R}, \mathfrak{R}'$ be as above. If $\phi(x_1, \ldots, x_n)$ is an $(L_{\mathfrak{R}})$-formula, and $h_1, \ldots, h_n \in H(\mathfrak{R})$, then*

$$\overline{H(\mathfrak{R})} \models \phi(h_1, \ldots, h_n) \Leftrightarrow \overline{H(\mathfrak{R}')} \models \phi(h_1, \ldots, h_n)$$

In other words the S-structure on $H(\mathfrak{R}')$ is an **elementary extension** *of $\overline{H(\mathfrak{R})}$.*

In particular the S-structures on $H(\mathfrak{R})$ and on $H(\mathfrak{R}')$ are **elementarily equivalent**.

Proof: By previous theorem, as if $f_1, \ldots, f_n$ are definable functions such that $[f_i] = h_i$, then

$$\overline{H(\mathfrak{R})} \models \phi(h_1, \ldots, h_n) \Leftrightarrow \exists \varepsilon > 0 : \forall t \in (0, \varepsilon) : \mathfrak{R} \models \phi(f_1(t), \ldots, f_n(t))$$

and

$$\overline{H(\mathfrak{R}')} \models \phi(h_1, \ldots, h_n) \Leftrightarrow \exists \varepsilon > 0 : \forall t \in (0, \varepsilon) : \mathfrak{R} \models \phi(f_1(t), \ldots, f_n(t))$$

$\square$

If $\mathfrak{R}$ is polynomially bounded, for every definable function $f : \mathbb{R}_{>0} \longrightarrow \mathbb{R}$ whose germ is not 0, there exists r in the field of exponents and $c \in \mathbb{R} \setminus \{0\}$ such that:

$$\lim_{x \to 0^+} \frac{f(x)}{x^r} = c$$

Both r and c are uniquely determined by the germ of f.

If h is the germ of f, we denote the exponent r by $v(h)$. The map $v : H(\mathfrak{R}) \setminus \{0\} \longrightarrow \mathbb{R}$ is a real valued **valuation**, turning $H(\mathfrak{R})$ in a **non-archimedean field** of rank one.

The image group of the valuation is the field of exponents of $\mathfrak{R}$, denoted by Λ. The valuation has a natural section, the map

$$\Lambda \ni r \longrightarrow x^r \in H(\mathfrak{R})$$

The **valuation ring**, denoted by $\mathcal{O}$, is the set of all germs bounded in a neighborhood of zero, and the **maximal ideal** m of $\mathcal{O}$ is the set of all germs infinitesimal in zero. The valuation ring $\mathcal{O}$ is **convex** with respect to the order $\leq$, hence the valuation topology coincides with the order topology. The map $\mathcal{O} \longrightarrow \mathbb{R}$ sending every element of $\mathcal{O}$ in its value in zero, has kernel m, hence it identifies in a natural way the residue field $\mathcal{O}/m$ with $\mathbb{R}$.

Example 64. *We will usually denote by $t \in H(\mathfrak{R})$ the germ of the identity function. We have $v(t) = 1$.*

1. *Every element of the field $H(\overline{\mathbb{R}})$ is algebraic over the fraction field $\mathbb{R}(t)$. Hence $H(\overline{\mathbb{R}})$ is the real closure of $\mathbb{R}(t)$, with reference to the unique order such that $t > 0$ and $\forall x \in \mathbb{R}_{>0} : t < x$. The image of the valuation is $\mathbb{Q}$. Consider the real closed field of formal Puiseaux series with real coefficients, $\mathbb{R}((t^{\mathbb{Q}})) = \bigcup_{n \geq 1} \mathbb{R}((t^{1/n}))$. The elements of this field have the form*

$$x^r(s(x^{1/n}))$$

   where $r \in \mathbb{Z}$ and s is a formal power series. As $\mathbb{R}(t) \subset \mathbb{R}((t^{\mathbb{Q}}))$ as an ordered field, then $H(\overline{\mathbb{R}}) \subset \mathbb{R}((t^{\mathbb{Q}}))$. The elements of $H(\overline{\mathbb{R}})$ are the elements of $\mathbb{R}((t^{\mathbb{Q}}))$ that are algebraic over $\mathbb{R}(t)$. For these elements the formal power series s is locally convergent.

2. *Consider the field $H(\mathbb{R}_{\text{an}}^{\Lambda})$ (see example 41). By [Mi94', cor. 2.7, prop. 4.5], for every element h of this field, there exist an analytic function $F : \mathbb{R}^n \longrightarrow \mathbb{R}$, a number $r \in \Lambda$ and numbers $r_1, \ldots, r_n \in \Lambda_{>0}$ such that*

$$h = [x^r F(x^{r_1}, \ldots, x^{r_n})]$$

   Vice versa, every element of this form is in $H(\mathbb{R}_{\text{an}}^{\Lambda})$.

3. *We have that $H(\overline{\mathbb{R}}^{\Lambda}) \subset H(\mathbb{R}^{\Lambda}_{an})$. Hence its elements have the same description.*

4. *Consider the field $H(\mathbb{R}_{\mathrm{an}^*})$ (see example 41). By [DS98, thm. B], for every element h of this field, exist a number $r \in \mathbb{R}$, a formal power series*
$$F = \sum_{\alpha \in \mathbb{R}_{\geq 0}} c_\alpha X^\alpha$$
and a radius $\delta \in \mathbb{R}_{>0}$ such that: $c_\alpha \in \mathbb{R}$, $\{\alpha \mid c_\alpha \neq 0\}$ is well ordered, the series $\sum_\alpha |c_\alpha| r^\alpha < +\infty$, (hence F is convergent and defines a continuous function on $[0, \delta]$, analytic on $(0, \delta)$) and
$$h = [x^r F(x)]$$

Let $\mathbb{F}$ be a real closed field extending $\mathbb{R}$. The convex hull of $\mathbb{R}$ in $\mathbb{F}$ is a valuation ring denoted by $\mathcal{O}_{\leq}$. This valuation ring defines a valuation $v : \mathbb{F}^* \longrightarrow \Lambda$, where Λ is an ordered abelian group. We say that $\mathbb{F}$ is a real closed **non-archimedean** field of **rank** one extending $\mathbb{R}$ if Λ has rank one as an ordered group, or, equivalently, if Λ is isomorphic to an additive subgroup of $\mathbb{R}$. Hence real closed non-archimedean fields of rank one extending $\mathbb{R}$ have a real valued valuation (non necessarily surjective) well defined up to a scaling factor. This valuation is well defined when we choose an element $t \in \mathbb{F}$ with $t > 0$ and $v(t) > 0$, and we choose a scaling factor such that $v(t) = 1$. Now a valuation $v : \mathbb{F} \longrightarrow \mathbb{R}$ is well defined, with image $v(\mathbb{F}^*) = \Lambda \subset \mathbb{R}$.

Consider the subfield $\mathbb{R}(t) \subset \mathbb{F}$. The order induced by $\mathbb{F}$ has the property that $t > 0$ and $\forall x \in \mathbb{R}_{>0} : t < x$. Hence $\mathbb{F}$ contains the real closure of $\mathbb{R}(t)$ with reference to this order, i.e. $H(\overline{\mathbb{R}})$. Moreover the valuation v on $\mathbb{F}$ restricts to the valuation we have defined on $H(\overline{\mathbb{R}})$, as, if $\mathcal{O}_{\leq}$ is the valuation ring of $\mathbb{F}$, $\mathcal{O}_{\leq} \cap H(\overline{\mathbb{R}})$ is precisely the valuation ring $\mathcal{O}$ of $H(\overline{\mathbb{R}})$. In other words every non-archimedean real closed field of rank one $\mathbb{F}$ extending $\mathbb{R}$ is a valued extension of $H(\overline{R})$. Note that $\mathbb{F}$ has an $\mathcal{OS}$-structure, $\overline{\mathbb{F}}$, as it is a real closed field, and $\overline{\mathbb{F}}$ is an elementary extension of $\overline{H(\overline{\mathbb{R}})}$, by the Tarski-Seidenberg principle (see [BCR98, prop. 5.2.3]).

4.1.2 Non archimedean amoebas

Let $\mathbb{F}$ be a non-archimedean real closed field of rank one extending $\mathbb{R}$, with a fixed real valued valuation $v : \mathbb{F}^* \longrightarrow \mathbb{R}$. By convention, we define $v(0) = \infty$, an element greater than any element of $\mathbb{R}$. The map
$$\mathbb{F} \ni h \longrightarrow \; \| h \| = \exp(-v(h)) \in \mathbb{R}_{\geq 0}$$
is a **non-archimedean norm**.

The component-wise **logarithm map** can be defined also on $\mathbb{F}$, by:
$$\mathrm{Log} : (\mathbb{F}_{>0})^n \ni (h_1, \ldots, h_n) \longrightarrow (\log(\| h_1 \|), \ldots, \log(\| h_n \|)) \in \mathbb{R}^n$$

Note that $\log(\| h \|) = -v(h)$. If $V \subset (\mathbb{F}_{>0})^n$, the **logarithmic image** of V is the image $\mathrm{Log}(V)$.

Let S be a set of symbols expanding $\mathcal{OS}$, and let $(\mathbb{F}, a)$ be an S-structure expanding the $\mathcal{OS}$-structure on the non-archimedean real closed field of rank one $\mathbb{F}$ extending $\mathbb{R}$. If $V \subset (\mathbb{F}_{>0})^n$ is a definable set in $(\mathbb{F}, a)$, we call the closure of the logarithmic image of V a **non-archimedean amoeba**, and we will write $\mathcal{A}(V) = \overline{\mathrm{Log}(V)}$.

The case we are more interested in is when $\mathfrak{R} = (\mathbb{R}, a)$ is an o-minimal, polynomially bounded S-structure expanding $\overline{\mathbb{R}}$, and $H(\mathfrak{R})$ is the Hardy field, with its natural valuation v and its natural S-structure. Non-archimedean amoebas of definable sets of $H(\mathfrak{R})$ are closely related with logarithmic limit sets of definable sets of $\mathfrak{R}$.

Let $\mathbb{F} \subset \mathbb{K}$ be two real closed fields. Let S be a set of symbols expanding $\mathcal{OS}$, let $(\mathbb{F}, a), (\mathbb{K}, b)$ be S-structures expanding the $\mathcal{OS}$ structure on the real closed fields and such that $\mathbb{K}$ is an elementary extension of $\mathbb{F}$. Let $V \subset \mathbb{F}^n$ be a definable set in $(\mathbb{F}, a)$. Choose an (L_S)-formula $\phi(x_1, \ldots, x_n, y_1, \ldots, y_m)$ and $a_1, \ldots, a_m \in \mathbb{R}$ such that

$$V = \{x \in \mathbb{F}^n \mid \phi(x_1, \ldots, x_n, a_1, \ldots, a_m)\}$$

We can define the **extension** $\overline{V}$ of V to $\mathbb{K}$ as

$$\overline{V} = \{x \in \mathbb{K}^n \mid \phi(x_1, \ldots, x_n, a_1, \ldots, a_m)\}$$

The extension is well defined. Suppose that there exist another (L_S)-formula $\psi(x_1, \ldots, x_n, y_1, \ldots, y_m)$ and $b_1, \ldots, b_l \in \mathbb{R}$ such that

$$V = \{x \in \mathbb{F}^n \mid \psi(x_1, \ldots, x_n, b_1, \ldots, b_l)\}$$

Then

$$(\mathbb{F}, a) \vDash \forall x_1 : \ldots \forall x_n : \phi(x_1, \ldots, x_n, a_1, \ldots, a_m) \Leftrightarrow \psi(x_1, \ldots, x_n, b_1, \ldots, b_l)$$

As $(\mathbb{K}, b)$ is an elementary extension, then

$$(\mathbb{K}, b) \vDash \forall x_1 : \ldots \forall x_n : \phi(x_1, \ldots, x_n, a_1, \ldots, a_m) \Leftrightarrow \psi(x_1, \ldots, x_n, b_1, \ldots, b_l)$$

Hence the extension defined by the formula ϕ coincide with the extension defined by the formula ψ.

For example, if $V \subset \mathbb{R}^n$ is a definable set in $\mathfrak{R}$, we can always define an extension of V to the Hardy field $H(\mathfrak{R})$. The extension will be denoted by $\overline{V} \subset H(\mathfrak{R})^n$.

Lemma 65. *Let $\mathfrak{R}$ be a o-minimal polynomially bounded structure. Let $V \subset (\mathbb{R}_{>0})^n$ be a definable set. Then*

$$(0, \ldots, 0, -1) \in \mathcal{A}_0(V) \Leftrightarrow (0, \ldots, 0, -1) \in \mathrm{Log}(\overline{V})$$

Proof : Suppose that $(-1, 0, \ldots, 0) \in \mathcal{A}(\overline{V})$. Then there is a point $(x_1, \ldots, x_n) \in \overline{V}$ such that $v(x_n) = 1$ and $v(x_i) = 0$ for all $i < n$. Then if $f_1, \ldots, f_n$ are definable functions such that $x_i = [f_i]$:

$$\exists \varepsilon > 0 : \forall t \in (0, \varepsilon) : (f_1(t), \ldots, f_n(t)) \in V$$

Moreover, when $t \to 0$ we have that $f_n(t) \to 0$ and $f_i(t) \to a_i > 0$ for $i < n$. Hence V contains a sequence tending to $(a_1, \ldots, a_{n-1}, 0)$ with $a_1, \ldots a_{n-1} \neq 0$, and $\mathcal{A}_0(V)$ contains $(0, \ldots, 0, -1)$.

Vice versa, suppose that $(0, \ldots, 0, -1) \in \mathcal{A}_0(V)$. Then, by theorem 47 there exists a sequence $x(k)$ in V such that $x(k) \to (a_1, \ldots, a_{n-1}, 0)$, where $a_1, \ldots, a_{n-1} > 0$. Let ε be a number less than all the numbers $a_1, \ldots, a_{n-1}$, and consider the set:

$$\{x \in \mathbb{R} \mid \exists x_1, \ldots, x_{n-1} : |x_i - a_i| < \tfrac{1}{2}\varepsilon \text{ and } (x_1, \ldots, x_{n-1}, x) \in V\}$$

As this set is definable, and as it contains a sequence converging to zero, it must contain an interval of the form $(0, \delta)$, with $\delta > 0$. In one formula:

$$\forall x \in (0, \delta) : \exists x_1, \ldots, x_{n-1} : |x_i - a_i| < \tfrac{1}{2}\varepsilon \text{ and } (x_1, \ldots, x_{n-1}, x) \in V$$

This sentence can be turned into a first order S-formula using a definition of V. This formula must also hold for $H(\mathfrak{R})$. We can choose an $x \in H(\mathfrak{R})$, with $x > 0$ and $v(x) = 1$. Then $x < \delta$, hence

$$\exists x_1, \ldots, x_{n-1} : |x_i - a_i| < \tfrac{1}{2}\varepsilon \text{ and } (x_1, \ldots, x_{n-1}, x) \in \overline{V}\}$$

Now $v(x_i) = 0$ for all $i > 1$, as $|x_i - a_i| < \frac{1}{2}\varepsilon$. Hence

$$\operatorname{Log}(x_1, \ldots, x_{n-1}, x) = (0, \ldots, 0, -1)$$

□

Theorem 66. *Let $\mathfrak{R}$ be a o-minimal polynomially bounded structure with field of exponents Λ. Let $V \subset (\mathbb{R}_{>0})^n$ be a definable set. Then*

$$\mathcal{A}_0(V) \cap \Lambda^n = \operatorname{Log}(\overline{V})$$

Proof : We need to prove that for all $x \in \Lambda^n$, $x \in \mathcal{A}_0(V) \Leftrightarrow x \in \mathcal{A}(\overline{V})$. We choose a matrix B with entries in Λ sending x in $(0, \ldots, 0, -1)$. Then we conclude by the previous lemma applied to the definable set $\overline{B}(V)$. □

Theorem 67. *Let $\mathbb{F} \subset \mathbb{K}$ be two non-archimedean real closed fields of rank one extending $\mathbb{R}$, with a choice of a real valued valuation defined by an element $t \in \mathbb{F}$. Denote the value groups by $\Lambda = v(\mathbb{F}^*)$ and $\Omega = v(\mathbb{K}^*)$. Let S be a set of symbols expanding $\mathcal{OS}$, let $(\mathbb{F}, a), (\mathbb{K}, b)$ be S-structures expanding the $\mathcal{OS}$ structure on the real closed fields and such that $\mathbb{K}$ is an elementary* *extension of* $\mathbb{F}$*. Let* V *be a definable set in* $(\mathbb{F}, a)$*, and* $\overline{V}$ *be its extension to* $(\mathbb{K}, b)$*. Then* $\mathrm{Log}(V) \subset \Lambda^n$ *is dense in* $\mathrm{Log}(\overline{V}) \subset \Omega^n$*.*

Proof : Suppose, by contradiction, that $x \in \mathrm{Log}(\overline{V})$ and it is not in the closure of $\mathrm{Log}(V)$. Then there exists an $\varepsilon > 0$ such that the cube

$$C = \{y \in \mathbb{R}^n \mid |y_1 - x_1| < \varepsilon, \dots, |y_n - x_n| < \varepsilon\}$$

does not contain points of $\mathrm{Log}(V)$.

Let $h \in \overline{V}$ be an element such that $\mathrm{Log}(h) = x$, and let $d \in \mathbb{F}$ be an element such that $0 < v(d) < \varepsilon$. Consider the cube

$$E = (\frac{h_1}{d}, h_1 d) \times (\frac{h_2}{d}, h_2 d) \times \cdots \times (\frac{h_n}{d}, h_n d) \subset \mathbb{K}^n$$

The image $\mathrm{Log}(E)$ is contained in C, hence $E \cap V$ is empty. But, as $(\mathbb{K}, b)$ is an elementary extension of $(\mathbb{F}, a)$, also $E \cap \overline{V}$ is empty. This is a contradiction as $h \in E$ and $h \in \overline{V}$. □

Corollary 68. *Let* S *be a set of symbols expanding* $\mathcal{OS}$*, and let* $\mathfrak{R} = (\mathbb{R}, a)$ *be an o-minimal polynomially bounded* S*-structure with field of exponents* Λ*, expanding* $\overline{\mathbb{R}}$*. Let* $V \subset (\mathbb{R}_{>0})^n$ *be a definable set. Suppose that there exists a subfield* $\Omega \subset \mathbb{R}$ *such that* $\Lambda \subset \Omega$ *and* $\mathfrak{R}^\Lambda$ *is o-minimal and polynomially bounded. Then* $\mathcal{A}_0(V) \cap \Lambda^n$ *is dense in* $\mathcal{A}_0(V) \cap \Omega^n$*.*

Proof : Consider the Hardy fields $H(\mathfrak{R})$ and $H(\mathfrak{R}^\Lambda)$. We denote by $\overline{V}$ the extension of V to $H(\mathfrak{R})$, and by $\overline{\overline{V}}$ the extension of V to $H(\mathfrak{R}^\Lambda)$. By theorem 66 $\mathcal{A}_0(V) \cap \Lambda^n = \mathrm{Log}(\overline{V})$ and $\mathcal{A}_0(V) \cap \Omega^n = \mathrm{Log}(\overline{\overline{V}})$. By corollary 63 the S-structures on $H(\mathfrak{R})$ and $H(\mathfrak{R}^\Lambda)$ are elementary equivalent. The statement follows by the previous theorem. □

Corollary 69. *Let* S *be a set of symbols expanding* $\mathcal{OS}$*, and let* $\mathfrak{R} = (\mathbb{R}, a)$ *be a regular polynomially bounded* S*-structure with field of exponents* Λ*. Let* $V \subset (\mathbb{R}_{>0})^n$ *be a set that is definable in* $\mathfrak{R}$*. We denote by* $\overline{V}$ *the extension of* V *to* $H(\mathfrak{R})$ *and by* $\overline{\overline{V}}$ *the extension of* V *to* $H(\mathfrak{R}^{\mathbb{R}})$*. Then*

$$\mathcal{A}_0(V) = \mathrm{Log}(\overline{\overline{V}})$$

Moreover the subset $\mathcal{A}_0(V) \cap \Lambda^n$ *is dense in* $\mathcal{A}_0(V)$*, and, as* $\mathcal{A}_0(V)$ *is closed,*

$$\mathcal{A}(\overline{V}) = \mathcal{A}(\overline{\overline{V}}) = \mathrm{Log}(\overline{\overline{V}})$$

□

Corollary 70. *Let* $V \subset (\mathbb{R}_{>0})^n$ *be a semi-algebraic set. Then* $\mathcal{A}_0(V) \cap \mathbb{Q}^n$ *is dense in* $\mathcal{A}_0(V)$*. Let* $\mathbb{F}$ *be a non-archimedean real closed field of rank one extending* $\mathbb{R}$*, and let* $\overline{V}$ *be the extension of* V *to* $\mathbb{F}$*. Then*

$$\mathcal{A}_0(V) = \mathcal{A}(\overline{V})$$

If $\mathbb{F}$ extends $H(\overline{\mathbb{R}}^{\mathbb{R}})$, then

$$\mathcal{A}(\overline{V}) = \mathrm{Log}(\overline{V})$$

$\square$

As a further corollary, we can now easily prove a proposition that was stated in the previous chapter.

Proposition 71. *Let $V \subset (\mathbb{R}_{>0})^n$ be a set definable in a regular polynomially bounded structure, and let $\pi : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be the projection on the first m coordinates (with $m < n$). Then we have*

$$\mathcal{A}_0(\pi(V)) = \pi(\mathcal{A}_0(V))$$

Proof: It follows easily from corollary 69 and from the fact that $\overline{\pi}(\overline{V}) = \overline{\pi(V)}$. $\square$

4.1.3 Patchworking families

Let S be a set of symbols expanding $\mathcal{OS}$, and let $\mathfrak{R} = (\mathbb{R}, a)$ be an S-structure expanding $\overline{\mathbb{R}}$. If $V \subset (H(\mathfrak{R})_{>0})^n$ is definable, there exists a first order S-formula $\phi(x_1, \ldots, x_n, y_1, \ldots, y_m)$, and parameters $a_1, \ldots, a_m \in H(\mathfrak{R})$ such that

$$V = \{(x_1, \ldots, x_n) \mid \phi(x_1, \ldots, x_n, a_1, \ldots, a_m)\}$$

Choose definable functions $f_1, \ldots, f_m$ such that $[f_i] = a_i$. These data defines a definable set in $\mathfrak{R}$:

$$\widetilde{V} = \{(x_1, \ldots, x_n, t) \in (\mathbb{R}_{>0})^{n+1} \mid \phi(x_1, \ldots, x_n, f_1(t), \ldots, f_m(t))\}$$

Suppose that $\phi'(x_1, \ldots, x_n, y_1, \ldots, y_{m'})$ is another formula defining V with parameters $a'_1, \ldots, a'_{m'}$, and that $f'_1, \ldots, f'_{m'}$ are definable functions such that $[f'_i] = a_i$. These data defines:

$$\widetilde{V}' = \{(x_1, \ldots, x_n, t) \in (\mathbb{R}_{>0})^{n+1} \mid \phi'(x_1, \ldots, x_n, f'_1(t), \ldots, f'_{m'}(t))\}$$

As both formulae defines V we have:

$$\overline{H(\mathfrak{R})} \models \forall x_1, \ldots, x_n : \phi(x_1, \ldots, x_n, a_1, \ldots, a_m) \Leftrightarrow \phi'(x_1, \ldots, x_n, a'_1, \ldots, a'_{m'})$$

By theorem 61 we have:

$$\exists \varepsilon > 0 : \forall t \in (0, \varepsilon) : \mathfrak{R} \models \forall x_1, \ldots, x_n :$$

$$\phi(x_1, \ldots, x_n, f_1(t), \ldots, f_m(t)) \Leftrightarrow \phi'(x_1, \ldots, x_n, f'_1(t), \ldots, f'_{m'}(t))$$

Hence

$$\widetilde{V} \cap (\mathbb{R}^n \times (0, \varepsilon)) = \widetilde{V}' \cap (\mathbb{R}^n \times (0, \varepsilon))$$

and the set $\widetilde{V}$ is "well defined for small enough values of t". Actually we prefer to see the set $\widetilde{V}$ as a parametrized family:

$$V_t = \{(x_1, \ldots, x_n) \in (\mathbb{R}_{>0})^n \mid (x_1, \ldots, x_n, t) \in \widetilde{V}\}$$

we can say that the set V determines the germ near zero of this parametrized family. We will use the notation $V_* = (V_t)_{t>0}$ for the family, and we will call these families **patchworking families** determined by V, as they are a generalization of the patchworking families of [Vi].

Given a patchworking family V_*, we can define the **tropical limit** of the family as:

$$\mathcal{A}_0(V_*) = \lim_{t \longrightarrow 0} \mathcal{A}_t(V_t) = \lim_{t \longrightarrow 0} \mathrm{Log}_{\frac{1}{t}}(V_t)$$

This is a closed subset of $\mathbb{R}^n$. Note that this set only depends on V. If V is the extension to $H(\mathfrak{R})$ of a definable subset $W \subset \mathbb{R}^n$, then the patchworking family V_t is constant: $V_t = W$, and the tropical limit is simply the logarithmic limit set: $\mathcal{A}_0(V_*) = \mathcal{A}_0(W)$.

Consider the logarithmic limit set of $\widetilde{V}$:

$$\mathcal{A}_0(\widetilde{V}) = \lim_{t \longrightarrow 0} \mathcal{A}_t(\widetilde{V}) \subset \mathbb{R}^{n+1}$$

As in subsection 3.2.2, we consider the set

$$H = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid (x_1, \ldots, x_n, -1) \in \mathcal{A}_0(\widetilde{V})\}$$

Note that

$$\mathrm{Log}^{-1}_{\left(\frac{1}{t}\right)}(\mathbb{R}^n \times \{-1\}) = (\mathbb{R}_{>0})^n \times \{t\}$$

Hence $\mathcal{A}_0(V_*) = \lim_{t \longrightarrow 0} \mathrm{Log}_{\frac{1}{t}}(V_t) = H$.

Now consider the extension of the set $\widetilde{V}$ to the Hardy field $H(\mathfrak{R})$, we denote it by $\overline{\widetilde{V}}$. By the results of the previous section, we know that $\mathcal{A}(\overline{\widetilde{V}}) = \mathcal{A}_0(\widetilde{V})$. If we denote by $t \in H(\mathfrak{R})$ the germ of the identity function, we have that

$$V = \{(x_1, \ldots, x_n) \mid (x_1, \ldots, x_n, t) \in \overline{\widetilde{V}}\}$$

as, for $i \in \{1, \ldots, m\}$, we have $f_i(t) = a_i$. Hence, as $\log|t| = -1$, $\mathcal{A}(V) \subset H = \mathcal{A}_0(V_*)$.

Lemma 72. $(0, \ldots, 0) \in \mathcal{A}_0(V_*) \Leftrightarrow (0, \ldots, 0) \in \mathrm{Log}(V)$.

Proof : $\Rightarrow$: This follows from what we said above.

$\Leftarrow$: It follows from the second part of the proof of lemma 65, applied to the set $\widetilde{V}$. □

Let $\lambda \in \Lambda^n$. We define a twisted set

$$V^\lambda = \{x \in H(\mathfrak{R})^n \mid \phi(t^{-\lambda_1}x_1, \ldots, t^{-\lambda_n}x_n, a_1, \ldots, a_m)\}$$

Then $\lambda \in \mathrm{Log}(V) \Leftrightarrow (0, \ldots, 0) \in \mathrm{Log}(V^\lambda)$. Then we define

$$H^\lambda = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid (x_1, \ldots, x_n, -1) \in \mathcal{A}_0(\widetilde{V^\lambda})\}$$

Now H^λ is simply H translated by the vector $-\lambda$. Hence we get the following:

Lemma 73. *For all $\lambda \in \Lambda$, we have $\lambda \in \mathcal{A}_0(V_*) \Leftrightarrow \lambda \in \mathrm{Log}(V)$.* □

Using these facts we can extend the results of the previous sections about logarithmic limit sets and their relations with non-archimedean amoebas, to tropical limits of patchworking families. For example we can prove the following statements.

Theorem 74. *Let S be a structure expanding $\mathcal{OS}$, and let $\mathfrak{R} = (\mathbb{R}, a)$ be a regular polynomially bounded S-structure with field of exponents Λ. Let V be a definable subset of the Hardy field $H(\mathfrak{R})$, and let V_* be a patchworking family determined by V. Then the following facts hold:*

1. *$\mathcal{A}_0(V_*)$ is a polyhedral complex with dimension less than or equal to the dimension of V.*
2. *$\mathcal{A}_0(V_*) \cap \Lambda^n = \mathrm{Log}(V)$.*
3. *$\mathcal{A}_0(V_*) = \mathcal{A}(V)$.*
4. *$\mathcal{A}_0(V_*) \cap \Lambda^n$ is dense in $\mathcal{A}_0(V_*)$.*

Proof: Every statement follows from the corresponding statement about logarithmic limit sets, and from the facts exposed above. □

For every point $\lambda \in \Lambda$, the twisted set V^λ defines a germ of patchworking family V_*^λ. The limit

$$V_0^\lambda = \lim_{t \to 0} V_t^\lambda$$

is a definable set in $\mathbb{R}^n$ and it has the properties of the set W of subsection 3.2.2. The difference is that now the set V_0^λ is well defined, and it depends only on λ.

Theorem 75. *Let S be a set of symbols expanding $\mathcal{OS}$, and let $\mathfrak{R} = (\mathbb{R}, a)$ be an S-structure expanding $\overline{R}$, that is o-minimal and polynomially bounded, with field of exponents Λ. Let V be a definable subset of the Hardy field $H(\mathfrak{R})$. Then we have*

$$\forall \lambda \in \Lambda^n : \lambda \in \mathcal{A}(V) \Leftrightarrow V_0^\lambda \neq \emptyset$$

Moreover, if $\Lambda = \mathbb{R}$, for all $\lambda \in \mathbb{R}$, there exists a neighborhood U of λ in $\mathcal{A}(V)$ such that the translation of U by $-\lambda$ is a neighborhood of $(0, \ldots, 0)$ in $\mathcal{A}_0(V_0^\lambda)$.

Proof: It follows from the arguments above and from theorem 52. □

The set V_0^λ can be called **initial set** of λ, as it plays the role of the initial ideal of [SS04]. The difference is that V_0^λ is a geometric object, while the initial ideal of [SS04] is a combinatorial one.

4.2 Comparison with other constructions

4.2.1 Complex algebraic sets

Logarithmic limit sets of complex algebraic sets are a particular case of logarithmic limit sets of real semi-algebraic sets, in the following sense. A finite set of polynomials $f_1, \ldots, f_m \in \mathbb{C}[x_1, \ldots, x_n]$ defines a complex algebraic set in $\mathbb{C}^n$:

$$V = V(f_1, \ldots, f_m) = \{x \in \mathbb{C}^n \mid f_1(x) = \cdots = f_m(x) = 0\}$$

Given a real number $\alpha > 1$, the map Log_α can be extended to the complex torus $(\mathbb{C}^*)^n$ by:

$$\mathrm{Log}_\alpha : (\mathbb{C}^*)^n \ni (z_1, \ldots, z_n) \longrightarrow (\log_\alpha(|z_1|), \ldots, \log_\alpha(|z_n|)) \in \mathbb{R}^n$$

Then for every $t \in (0,1)$ the amoeba of V is defined as

$$\mathcal{A}_t(V) = \mathrm{Log}_{\frac{1}{t}}(V \cap (\mathbb{C}^*)^n)$$

and the logarithmic limit set can be defined as

$$\mathcal{A}_0(V) = \lim_{t \to 0} \mathcal{A}_t(V)$$

Another way for giving this definition is the following: consider the real semi-algebraic set

$$|V| = \{x \in (\mathbb{R}_{>0})^n \mid \exists z \in V : |z| = x\}$$

Proposition 76. *The logarithmic limit set of the complex algebraic set V is equal to the logarithmic limit set of the real semi-algebraic set $|V|$.* □

Hence all the results we got about logarithmic limit sets of real semi-algebraic sets produce an alternative proof of the same results for complex algebraic sets.

Let $x \in \mathbb{Q}^n$. Up to multiplying x by a positive rational number, we can suppose that x is a primitive integer vector. Then we choose a map $B \in GL_n(\mathbb{Z})$ such that $B(x) = (0, \ldots, 0, -1)$. The map $\overline{B}$ only contains ordinary integer exponents, hence can be extended to a map $\overline{B} : (\mathbb{C}^*)^n \longrightarrow (\mathbb{C}^*)^n$, and as $B \in GL_n(\mathbb{Z})$, $\overline{B}$ is an automorphism of the complex torus. The set $\overline{B}(V)$ is again a complex algebraic set and $\overline{B}(|V|) = |\overline{B}(V)|$. In subsection 3.2.2, for every $t \in (0,1)$, we defined the set:

$$W_t = \{(x^1, \ldots, x^{n-1}) \in (\mathbb{R}_{>0})^{n-1} \mid (x^1, \ldots, x^{n-1}, t) \in |\overline{B}(V)|\}$$

And the limit:

$$W = \left(\lim_{t \to 0} W_t\right) \cap (\mathbb{R}_{>0})^{n-1}$$

Note that the complex algebraic set

$$W^{\mathbb{C}} = \{(z^1, \dots, z^{n-1}) \in \mathbb{C}^{n-1} \mid (z_1, \dots, z^{n-1}, 0) \in V\}$$

has the property that $W = |W^{\mathbb{C}}|$. By the results of subsection 3.2.2, $x \in \mathcal{A}_0(V)$ if and only if $W^{\mathbb{C}} \cap (\mathbb{C}^*)^n \neq \emptyset$, and a neighborhood of x in $\mathcal{A}_0(V)$ is isomorphic to the cone over a neighborhood of 0 in $\mathcal{A}_0(W^{\mathbb{C}})$. All this lead to an alternative proof of the following theorem, originally proved partly in [Be71] and partly in [BG81].

Corollary 77. *Let $V \subset \mathbb{C}^n$ be a complex algebraic set of dimension d. Then $\mathcal{A}_0(V) \cap \mathbb{Q}^n$ is dense in $\mathcal{A}_0(V)$, $\mathcal{A}_0(V)$ is a polyhedral complex of dimension d, and if all the irreducible components of V have the same dimension, then every maximal face of $\mathcal{A}_0(V)$ has dimension d.*

Proof : Everything but the statement about dimension follows directly from the corresponding theorems about logarithmic limit sets of semi-algebraic sets. For the dimension note that if V is irreducible, then $W^{\mathbb{C}} \cap (\mathbb{C}^*)^n$ is either empty or a complex algebraic set of complex dimension one less than the dimension of V. We can conclude by an inductive argument analogous to the one used in the proof of theorem 53. □

Let $f \in \mathbb{R}[x_1, \dots, x_n]$. Let V be the intersection of the zero locus of f and $(\mathbb{R}_{>0})^n$, and let $V_{\mathbb{C}}$ be the zero locus of f in $\mathbb{C}^n$. As $V \subset V_{\mathbb{C}}$, the logarithmic limit set of V is included in the logarithmic limit set of $V_{\mathbb{C}}$. Moreover, as $V_{\mathbb{C}}$ is a complex hypersurface, it is possible to give an easy combinatorial description of $\mathcal{A}_0(V_{\mathbb{C}})$, it is simply the dual fan of the newton polytope of f. Unfortunately, it is not possible, in general, to use this fact to understand the combinatorics of $\mathcal{A}_0(V)$. There are examples where V is an irreducible hypersurface, and $\mathcal{A}_0(V)$ is a subpolyhedron of $\mathcal{A}_0(V_{\mathbb{C}})$ that is not a subcomplex. For example, if f is as in figure 4.1, the logarithmic limit set of V is only the ray in the direction $(-1, 0, 0)$, but this ray lies in the interior of a face of $\mathcal{A}_0(V_{\mathbb{C}})$.

Even the description of logarithmic limit sets via non-archimedean amoebas can be translated to complex algebraic sets. Let $\mathbb{F}$ be a non-archimedean real closed field of rank one extending $\mathbb{R}$, and let v be a choice of a real valued valuation on $\mathbb{F}$. The field $\mathbb{K} = \mathbb{F}[i]$ is an algebraically closed field extending $\mathbb{C}$, with an extended valuation $v : \mathbb{K}^* \longrightarrow \mathbb{R}$ defined by $v(a + bi) = \min(v(a), v(b))$. This extended valuation also extends the non-archimedean norm to $\mathbb{K}$ via

$$\mathbb{K} \ni z \longrightarrow \parallel z \parallel = \exp(-v(z)) \in \mathbb{R}_{\geq 0}$$

The component-wise **logarithm map** can be extended to $\mathbb{K}$, by:

$$\mathrm{Log} : \mathbb{K}^n \ni (z_1, \dots, z_n) \longrightarrow (\log(\parallel z_1 \parallel, \dots, \log \parallel z_n \parallel) \in \mathbb{R}^n$$

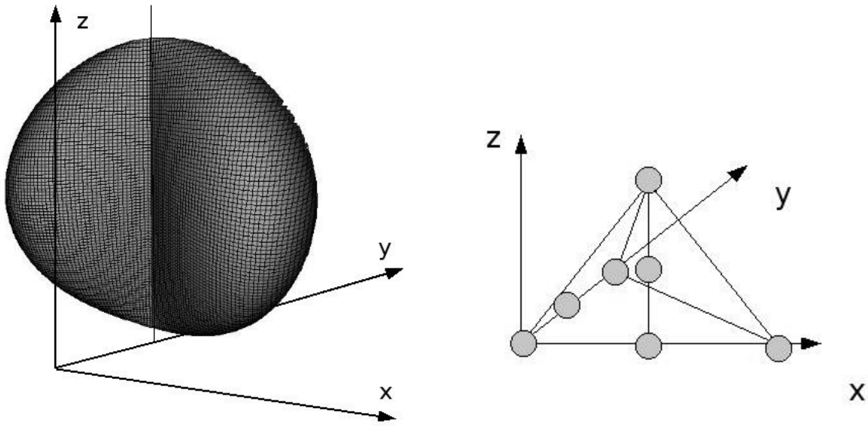

Figure 4.1: $V = \{(x, y, z) \in \mathbb{R}^3 \mid x^2(1-(z-2)^2) = x^4 + (y-1)^2\}$, it is an irreducible surface, but it has a "stick", the line $\{y = 1, x = 0\}$. The logarithmic limit set of $V \cap (\mathbb{R}_{>0})^3$ is only the ray in the direction $(-1, 0, 0)$, but this ray is contained in the interior part of a face of the dual fan of the newton polytope of the defining polynomial $x^2(1-(z-2)^2) - x^4 - (y-1)^2$. Picture from [BCR98].

As before $\log(\parallel z \parallel) = -v(z)$. On $\mathbb{K}$ there is also the **complex norm** $|\cdot| : \mathbb{K} \longrightarrow \mathbb{F}_{\geq 0}$ defined by $|a + bi| = \sqrt{a^2 + b^2}$. The two norms are related by

$$\forall z \in \mathbb{K} : \parallel |z| \parallel = \parallel z \parallel$$

Now if V is an algebraic set in $\mathbb{K}^n$, the set

$$|V| = \{x \in (\mathbb{F}_{>0})^n \mid \exists z \in V : |z| = x\}$$

is a semi-algebraic set in $\mathbb{F}^n$. The **logarithmic image** of V is the image $\mathrm{Log}(V)$, and the non-archimedean amoeba $\mathcal{A}(V)$ is the closure of this image. As expected, $\mathrm{Log}(V) = \mathrm{Log}(|V|)$ and $\mathcal{A}(V) = \mathcal{A}(|V|)$. Moreover, if $V \subset \mathbb{C}^n$ is an algebraic set, and $\overline{V} \subset \mathbb{K}^n$ is its extension to $\mathbb{K}$, then $\overline{|V|} = |\overline{V}|$.

These facts directly give the relation between logarithmic limit sets of complex algebraic sets and non-archimedean amoebas in algebraically closed fields.

Corollary 78. *Let $V \subset \mathbb{C}^n$ be an algebraic set. Let $\mathbb{F}$ be a non-archimedean real closed field of rank one extending $\mathbb{R}$, and let $\overline{V}$ be the extension of V to $\mathbb{K} = \mathbb{F}[i]$. Then*

$$\mathcal{A}_0(V) = \mathcal{A}(\overline{V})$$

If $\mathbb{F}$ extends the Hardy field $H(\overline{\mathbb{R}}^{\mathbb{R}})$, then

$$\mathcal{A}(\overline{V}) = \mathrm{Log}(\overline{V})$$

□

Now let $\mathfrak{R} = (\mathbb{R}, a)$ be a regular polynomially bounded structure with field of exponents Λ. Let $\mathbb{K} = H(\mathfrak{R})[i]$ and let $V \subset \mathbb{K}^n$ be an algebraic set. There are polynomials $f_1, \ldots, f_m \in \mathbb{K}[x_1, \ldots, x_n]$ such that $V = V(f_1, \ldots, f_m)$. Every polynomial f_j has the form

$$f_j = \sum_{\omega \in \mathbb{Z}^n} (a_{j,\omega} + ib_{j,\omega}) x^\omega$$

where ω is a multi-index, the set $S(f) = \{\omega \in \mathbb{Z}^n \mid a_\omega + ib_\omega \neq 0\}$ is finite and $a_{j,\omega}, b_{j,\omega} \in H(\mathfrak{R})$. Choose functions $\alpha_{j,\omega}, \beta_{j,\omega}$ that are definable in $\mathfrak{R}$ and such that $[\alpha_{j,\omega}] = a_{j,\omega}, [\beta_{j,\omega}] = b_{j,\omega}$. This choice defines families of polynomials

$$f_{j,t} = \sum_{\omega \in S(f)} (\alpha_{j,\omega}(t) + i\beta_{j,\omega}(t)) x^\omega$$

and a corresponding family of algebraic sets in $\mathbb{C}^n$

$$V_t = V(f_{1,t}, \ldots, f_{m,t})$$

We will call these families **patchworking families** because they generalize the patchworking polynomial of [Mi, Part 2], and we will denote the family by $V_* = (V_t)$. The family V_* depends of the choice of the polynomials f_j and of the definable functions $\alpha_{j,\omega}, \beta_{j,\omega}$. If we change these choices we get another patchworking family coinciding with V_* for $t \in (0, \varepsilon)$. The **tropical limit** of one such family is

$$\mathcal{A}_0(V_*) = \lim_{t \longrightarrow 0} \mathcal{A}_t(V_t)$$

As before, $|V|$ is a semi-algebraic set in $H(\mathfrak{R})^n$, and if $|V|_* = (|V|_t)$ is a patchworking family defined by $|V|$, then there exists an $\varepsilon > 0$ such that for $t \in (0, \varepsilon)$ we have $|V_t| = |V|_t$. Hence we have that

$$\mathcal{A}_0(V_*) = \mathcal{A}_0(|V|_*)$$

and we can get the properties of the tropical limit of complex patchworking families as a corollary of the properties of tropical limits of real patchworking families:

Corollary 79. *The following facts hold:*

1. *$\mathcal{A}_0(V_*)$ is a polyhedral complex with dimension less than or equal to the dimension of V.*

2. *$\mathcal{A}_0(V_*) \cap \Lambda^n = \mathrm{Log}(V)$.*

3. *If $\mathfrak{R}'$ is an expansion of $\mathfrak{R}$, o-minimal, polynomially bounded, with field of exponents $\Omega \supset \Lambda$, and $\overline{V}$ is the extension of V to $H(\mathfrak{R}')[i]$, then* $\mathrm{Log}(V)$ *is dense in* $\mathrm{Log}(\overline{V})$*, and in particular $\mathcal{A}_0(V_*) \cap \Lambda^n$ is dense in $\mathcal{A}_0(V_*) \cap \Omega^n$.*

4. $\mathcal{A}(V) = \mathcal{A}_0(V_*)$

□

4.2.2 Positive tropical varieties

In this subsection we compare the notion of non-archimedean amoebas for real closed fields that we studied in this chapter with a similar object called positive tropical variety studied in [SW].

To be consistent with [SW], we will denote by $\mathbb{K} = \bigcup_{n=1}^{\infty} \mathbb{C}((t^{1/n}))$ the algebraically closed field of formal Puiseux series with complex coefficients, whose set of exponents is an arithmetic progression of rational numbers, and by $\mathbb{F} = \bigcup_{n=1}^{\infty} \mathbb{R}((t^{1/n}))$ the subfield of series with real coefficients. $\mathbb{K}$ is the algebraic closure of $\mathbb{F}$. These fields have a natural valuation $v : \mathbb{K} \longrightarrow \mathbb{Q}$, with valuation ring $\mathcal{O}$, and residue map $r : \mathcal{O} \longrightarrow \mathbb{C}$. Note that the valuation v is compatible with the order of $\mathbb{F}$, i.e. the valuation ring $\mathcal{O} \cap \mathbb{F}$ is convex for the order, and that $r(\mathcal{O} \cap \mathbb{F}) = \mathbb{R}$.

We will denote by $\mathbb{F}_{>0}$ the set of positive elements of the field $\mathbb{F}$. Following [SW] we will also use the notation:

$$\mathbb{F}_+ = \{z \in \mathbb{K} \mid r\left(\frac{z}{t^{v(z)}}\right) \in \mathbb{R}_{>0}\}$$

Let V be an algebraic set in $\mathbb{K}^n$. The set

$$V_{>0} = V \cap (\mathbb{F}_{>0})^n$$

is a semi-algebraic set, whose non-archimedean amoeba $\mathcal{A}(V_{>0})$ (i.e. the closure of the logarithmic image $\mathrm{Log}(V_{>0})$) has been studied in subsection 4.1.3. In [SW] a similar definition is given. The positive part of V is defined as

$$V_+ = V \cap (\mathbb{F}_+)^n$$

The closure of $\mathrm{Log}(V_+)$ is called **positive tropical variety**, and it is denoted by $\mathrm{Trop}^+(V)$. From the definition it is clear that $\mathcal{A}(V_{>0}) \subset \mathrm{Trop}^+(V)$.

In many examples the sets $\mathcal{A}(V_{>0})$ and $\mathrm{Trop}^+(V)$ coincide, but it is also possible to construct examples where the inclusion is strict. Consider the set

$$V = \{z \in \mathbb{K}^2 \mid x_1^2 + (x_2 - 1)^2 - x_1^3\}$$

Then $V_{>0}$ is the extension to $\mathbb{F}$ of the set in figure 3.7.

$$\mathcal{A}(V_{>0}) = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 = 0, x_2 \leq 0 \text{ or } x_1 \geq 0, 2x_2 = 3x_1\}$$

$$\mathrm{Trop}^+(V) = \mathcal{A}(V_{>0}) \cup \{x_2 = 0, x_1 \leq 0\}$$

A more interesting example where $\mathcal{A}(V_{>0}) \subsetneq \mathrm{Trop}^+(V)$ is the set

$$V = \{(x, y, z) \in \mathbb{K}^3 \mid x^2(1 - (z-2)^2) = x^4 + (y-1)^2\}$$

Here $V_{>0}$ is the extension to $\mathbb{F}$ of the set in figure 4.1. Now $\mathcal{A}(V_{>0})$ is just the ray in the direction $(-1, 0, 0)$. This ray is in the interior part of a face of $\mathrm{Trop}^+(V)$. Hence not only the two sets does not coincide, but $\mathcal{A}(V_{>0})$ is not a polyhedral subcomplex of $\mathrm{Trop}^+(V)$.

Chapter 5

Compactification

In this chapter, we apply the Maslov dequantization to the Teichmüller spaces and to the spaces of convex projective structures. More precisely, we construct an inverse systems of logarithmic limit sets, whose inverse limit we use to construct a compactification. This inverse limit is a cone, and it is the image of the extension of the space to a real closed non-archimedean field under a componentwise valuation map. For the Teichmüller spaces, the boundary constructed in this way is the Thurston boundary, endowed with a natural piecewise linear structure, that is equivalent to the one defined by Thurston. This shows how the piecewise linear structure on the boundary is induced by the semi-algebraic structure on the interior part. With the same techniques we construct a compactification for the spaces of convex projective structures on a closed n-manifold.

This construction is a generalization of the Morgan-Shalen compactification (see [MS84]). They worked in the framework of complex algebraic geometry: given a complex variety V and countable family of polynomial functions $\{f_i\}_{i\in I}$ containing a set of generators of the ring of coordinates of V, they constructed a compactification of V taking the limit points of the ratios of the values $[\log(|f_i(x)|+2)]_{i\in I}$. Here we work directly with real semi-algebraic sets, and this allows us to consider more general families of functions. Given a semi-algebraic set V and a proper family of positive continuous semi-algebraic functions $\{f_i\}_{i\in I}$, we can construct a compactification of V, in a way that is shown to be equivalent to taking the limit points of the ratios of the values $[\log(f_i(x))]_{i\in I}$. The properties of the Morgan-Shalen compactification can be extended to this more general setting. This generalization is important for the construction of the compactification for the spaces of convex projective structures on a closed n-manifold, as the family of functions we consider there is not a family of polynomial functions.

A compactification construction related to this one is the one in [Te], called tropical compactification. The two compactifications are different, for example the tropical compactification is a complex variety and the boundary is a divisor, while the compactifications we present here don't have a structure of algebraic variety, here the objects we put on the boundary have a polyhedral nature. Anyway the two notions are related, as we show in subsection 5.2.2.

5.1 Compactification of semi-algebraic sets

5.1.1 Embedding-dependent compactification

Let $V \subset (\mathbb{R}_{>0})^n$ be a closed semi-algebraic set. We can construct a compactification for V, using its logarithmic limit set $\mathcal{A}_0(V) \subset \mathbb{R}^n$, a polyhedral fan.

This fan represents the behavior at infinity of the amoeba, hence it can be used to compactify it. We take the quotient by the spherical equivalence relation

$$x \sim y \Leftrightarrow \exists \lambda > 0 : x = \lambda y$$

and we get the boundary

$$\partial V = (\mathcal{A}_0(V) \setminus \{0\})/\sim \ \subset S^{n-1}$$

Now we glue ∂V to V at infinity in the following way. We compactify $\mathbb{R}^n$ by adding the sphere at infinity:

$$\mathbb{R}^n \ni x \longrightarrow \frac{x}{\sqrt{1+\|x\|^2}} \in D^n \qquad\qquad D^n \approx \mathbb{R}^n \cup S^{n-1}$$

Given a $t_0 < 1$, we will denote by $\overline{V}$ the closure of $\mathcal{A}_{t_0}(V)$ in D^n. Then

$$\overline{V} = \mathcal{A}_{t_0}(V) \cup \partial V$$

Proposition 80. *The map* $\mathrm{Log}_{\left(\frac{1}{t_0}\right)} : V \longrightarrow \overline{V}$ *is a compactification of* V*. The compactification does not depend on the choice of* t_0*.*

Proof : The map $\mathrm{Log}_{\left(\frac{1}{t_0}\right)}$ is a homeomorphism between V and $\mathcal{A}_{t_0}(V)$. The set $\mathcal{A}_{t_0}(V)$ is closed in $\mathbb{R}^n$, hence its closure is the union of $\mathcal{A}_{t_0}(V)$ and a subset of S^{n-1}. As S^{n-1} is closed in D^n, $\mathcal{A}_{t_0}(V)$ is open and dense in $\overline{V}$. □

Note that the logarithmic limit set $\mathcal{A}_0(V)$ is the cone over the boundary, and for this reason it will sometimes be denoted by $C(\partial V)$.

5.1.2 Embedding-independent compactification

This construction can be generalized in a way that does not depend on the immersion of V in $\mathbb{R}^n$. Let $V \subset \mathbb{R}^n$ be a semi-algebraic set. A finite family of continuous semi-algebraic functions $\mathcal{F} = \{f_1, \ldots, f_m\}$, with $f_i : V \longrightarrow \mathbb{R}_{>0}$, is called a **proper family** if the map

$$E_{\mathcal{F}} : V \ni x \longrightarrow (f_1(x), \ldots, f_m(x)) \in (\mathbb{R}_{>0})^m$$

is proper. In this case the map $L_{\mathcal{F}} = \mathrm{Log}_{\left(\frac{1}{t_0}\right)} \circ E_{\mathcal{F}}$ is also proper.

The image $E_{\mathcal{F}}(V) \subset (\mathbb{R}_{>0})^n$ is a closed semi-algebraic subset, and we can compactify it as before, by $\overline{E_{\mathcal{F}}(V)} = \mathcal{A}_{t_0}(E_{\mathcal{F}}(V)) \cup \partial E_{\mathcal{F}}(V)$.

Let $\hat{V} = V \cup \{\infty\}$ denote the Alexandrov compactification of V. Consider the map

$$i : V \ni x \longrightarrow (x, L_{\mathcal{F}}(x)) \in \hat{V} \times \overline{E_{\mathcal{F}}(V)}$$

and let $\overline{V}_{\mathcal{F}}$ be the closure of the image $i(V)$ in $\hat{V} \times \overline{E_{\mathcal{F}}(V)}$.

Proposition 81. *The map $i : V \longrightarrow \overline{V}_{\mathcal{F}}$ is a compactification of V. The boundary $\partial_{\mathcal{F}} V = \overline{V}_{\mathcal{F}} \setminus i(V)$ is the set $\partial E_{\mathcal{F}}(V)$.*

Proof : The image of i is homeomorphic to V (the inverse being the projection p_1 on the first factor). The space $\overline{V}_{\mathcal{F}}$ is compact as it is closed in a compact. The complement of $i(V)$ in $\overline{V}_{\mathcal{F}}$ is the set $p_1^{-1}(\infty) \cap \overline{V}_{\mathcal{F}}$, a closed set. Hence the map $i : V \longrightarrow \overline{V}_{\mathcal{F}}$ is a topological immersion of V in a open dense subset of a compact space, i.e. a compactification.

The boundary is $\partial_{\mathcal{F}} V = p_1^{-1}(\infty) \cap \overline{V}_{\mathcal{F}}$. The projection p_2 on the second factor identifies $\partial_{\mathcal{F}} V$ with a subset of $\overline{E_{\mathcal{F}}(V)}$. As the map $E_{\mathcal{F}}$ is proper, we have $\partial_{\mathcal{F}} V = \partial E_{\mathcal{F}}(V)$. $\square$

The cone over the boundary will be denoted by $C(\partial_{\mathcal{F}} V) = \mathcal{A}_0(E_{\mathcal{F}}(V))$.

5.1.3 Limit compactification

Here we present a further generalization of the construction of the compactification. This generalization is needed if we want to extend the action of a group on the semi-algebraic set to an action on the compactification, as in subsection 5.2.1.

Let $V \subset \mathbb{R}^n$ be a semi-algebraic set. A (possibly infinite) family of continuous semi-algebraic functions $\mathcal{G} = \{f_i\}_{i \in I}$, with $f_i : V \longrightarrow \mathbb{R}_{>0}$, is called a **proper family** if there exist a finite subfamily $\mathcal{F} \subset \mathcal{G}$ that is proper.

Suppose that $\mathcal{G}$ is proper. Let

$$P_{\mathcal{G}} = \{\mathcal{F} \subset \mathcal{G} \mid \mathcal{F} \text{ is proper } \}$$

a non-empty set partially ordered by inclusion. If $\mathcal{F} \subset \mathcal{F}'$ we denote by $\pi_{\mathcal{F}', \mathcal{F}}$ the projection

$$\pi_{\mathcal{F}', \mathcal{F}} : \mathbb{R}^{\mathcal{F}'} \longrightarrow \mathbb{R}^{\mathcal{F}}$$

on the coordinates corresponding to $\mathcal{F}$. This projection restricts to a surjective map

$$\pi_{\mathcal{F}',\mathcal{F}|\mathcal{A}_{t_0}(E_{\mathcal{F}'}(V))} : \mathcal{A}_{t_0}(E_{\mathcal{F}'}(V)) \longrightarrow \mathcal{A}_{t_0}(E_{\mathcal{F}}(V))$$

By proposition 71, the restriction to the logarithmic limit sets is also surjective:

$$\pi_{\mathcal{F}',\mathcal{F}|\mathcal{A}_0(E_{\mathcal{F}'}(V))} : \mathcal{A}_0(E_{\mathcal{F}'}(V)) \longrightarrow \mathcal{A}_0(E_{\mathcal{F}}(V))$$

Proposition 82. *Let $\mathcal{F}, \mathcal{F}' \in P_{\mathcal{G}}$. If $\mathcal{F} \subset \mathcal{F}'$, the map $\pi_{\mathcal{F}',\mathcal{F}|\mathcal{A}_0(E_{\mathcal{F}'}(V))}$ induces a map*

$$\partial\pi_{\mathcal{F}',\mathcal{F}} : \partial_{\mathcal{F}'}V \longrightarrow \partial_{\mathcal{F}}V$$

Proof : We have to prove that

$$\left(\pi_{\mathcal{F}',\mathcal{F}|\mathcal{A}_0(E_{\mathcal{F}'}(V))}\right)^{-1}(0) = \{0\}$$

Let $x \in \mathcal{A}_0(E_{\mathcal{F}'}(V)) \setminus \{0\}$, hence the set $\widetilde{\mathcal{F}'} = \{f \in \mathcal{F}' \mid x_f \neq 0\}$ is not empty. The image of x in the quotient $\partial_{\mathcal{F}'}(V)$ is the point $[x]$. As $\overline{V}_{\mathcal{F}'}$ is a compactification, there exists a sequence $(x_n) \subset V$ such that $(x_n) \to [x]$ in $\overline{V}_{\mathcal{F}'}$. As in section 3.2 we can choose (x_n) such that, for all function $f \in \mathcal{F}' \setminus \widetilde{\mathcal{F}'}$, the sequence $\log_{\left(\frac{1}{t_0}\right)}(f(x_n))$ is bounded. As the map $L_{\mathcal{F}}$ is proper, the sequence $(L_{\mathcal{F}}(x_n))$ is not bounded, hence there is a function $f \in \mathcal{F}$ such that $\log_{\left(\frac{1}{t_0}\right)}(f(x_n))$ is unbounded. Hence the corresponding coordinate $x_f \neq 0$, and $\pi_{\mathcal{F}',\mathcal{F}}(x) \neq 0$. □

The maps $\pi_{\mathcal{F}',\mathcal{F}}$ and $\partial\pi_{\mathcal{F}',\mathcal{F}}$ define inverse systems $\{\mathcal{A}_{t_0}(E_{\mathcal{F}}(V))\}_{\mathcal{F}\in P_{\mathcal{G}}}$, $\{\mathcal{A}_0(E_{\mathcal{F}}(V))\}_{\mathcal{F}\in P_{\mathcal{G}}}$ and $\{\partial_{\mathcal{F}}V\}_{\mathcal{F}\in P_{\mathcal{G}}}$. Consider the inverse limit

$$L = \varprojlim \mathcal{A}_{t_0}(E_{\mathcal{F}}(V))$$

we will denote by $\pi_{\mathcal{G},\mathcal{F}} : L \longrightarrow \mathcal{A}_{t_0}(E_{\mathcal{F}}(V))$ the canonical projection. By the explicit description of the inverse limit, L is a closed subset of the product:

$$\left\{ (x_{\mathcal{F}}) \in \prod_{\mathcal{F}\in P_{\mathcal{G}}} \mathcal{A}_{t_0}(E_{\mathcal{F}}(V)) \mid \forall \mathcal{F} \subset \mathcal{F}' : \pi_{\mathcal{F}',\mathcal{F}}(x_{\mathcal{F}'}) = x_{\mathcal{F}} \right\}$$

For every $x \in L$, and every $f \in \mathcal{G}$, let $\mathcal{F}$ be a proper finite family containing f. Then the value of the f-coordinate of the point $\pi_{\mathcal{G},\mathcal{F}}(x)$ does not depend on the choice of the family $\mathcal{F}$. This value will be denoted by x_f. The map

$$L \ni x \longrightarrow (x_f)_{f\in\mathcal{G}} \in \mathbb{R}^{\mathcal{G}}$$

identifies L with a subset of $\mathbb{R}^{\mathcal{G}}$.

The system of maps $L_{\mathcal{F}} : V \longrightarrow \mathcal{A}_{t_0}(E_{\mathcal{F}}(V))$, defined for every $\mathcal{F} \in P_{\mathcal{G}}$, induces by the universal property a well defined map $L_{\mathcal{G}} : V \longrightarrow L$.

Proposition 83. *The map $L_{\mathcal{G}}$ is surjective and proper, and it can be identified with the map*

$$V \ni x \longrightarrow \left(\log_{\left(\frac{1}{t_0}\right)}(f(x))\right)_{f\in\mathcal{G}} \in \mathbb{R}^{\mathcal{G}}$$

Proof : We proceed by transfinite induction on the cardinality of $\mathcal{G}$. If $\mathcal{G}$ is finite, the statement is trivial, as in this case $\mathcal{G}$ is the maximum of $P_{\mathcal{G}}$, the inverse limit is simply $L = \mathcal{A}_{t_0}(E_{\mathcal{G}}(V))$ and we now that for $\mathcal{F} \in P_{\mathcal{G}}$ the maps $L_{\mathcal{F}} : V \longrightarrow \mathcal{A}_{t_0}(E_{\mathcal{F}}(V))$ are surjective and proper.

Now suppose, by induction, that the statement is true for all proper families with cardinality less than the cardinality of $\mathcal{G}$. We denote by $P'_{\mathcal{G}}$ the set of all proper subfamilies of $\mathcal{G}$ with smaller cardinality. Let $y \in L$. By the inductive hypothesis for every $\mathcal{F} \in P'_{\mathcal{G}}$, the map $L_{\mathcal{F}} : V \longrightarrow \mathcal{A}_{t_0}(E_{\mathcal{F}}(V))$ is surjective and proper, hence the inverse image $L_{\mathcal{F}}^{-1}(\pi_{\mathcal{G},\mathcal{F}}(y))$ is a compact and non-empty subset of V.

By the Zermelo theorem every set has a cardinal well ordering, i.e. a well ordering such that all initial segments have cardinality less than the cardinality of the set. Moreover we can choose a finite set that will be an initial segment for the well ordering. We choose a cardinal well ordering $\prec$ of $\mathcal{G}$ such that a finite proper family $\overline{\mathcal{F}}$ is an initial segment. Consider the set $Q_{\mathcal{G}}$ of all initial segments of $\mathcal{G}$ containing $\overline{\mathcal{F}}$.

We have $Q_{\mathcal{G}} \subset P'_{\mathcal{G}}$, hence for every $\mathcal{F} \in Q_{\mathcal{G}}$ the subset $L_{\mathcal{F}}^{-1}(\pi_{\mathcal{G},\mathcal{F}}(y)) \subset V$ is compact and non-empty, and if $\mathcal{F} \subset \mathcal{F}'$, then $L_{\mathcal{F}}^{-1}(\pi_{\mathcal{G},\mathcal{F}}(y)) \subset L_{\mathcal{F}'}^{-1}(\pi_{\mathcal{G},\mathcal{F}'}(y))$. The sets $L_{\mathcal{F}}^{-1}(\pi_{\mathcal{G},\mathcal{F}}(y))$ are nested compact subsets of V, hence their intersection is non-empty:

$$\bigcap_{\mathcal{F}\in Q_{\mathcal{G}}} L_{\mathcal{F}}^{-1}(\pi_{\mathcal{G},\mathcal{F}}(y)) \neq \emptyset$$

If x is a point in this intersection, then $L_{\mathcal{G}}(x) = y$. The fact that the map $L_{\mathcal{G}}$ can be identified with the map $\left(\log_{\left(\frac{1}{t_0}\right)}(f(x))\right)_{f\in\mathcal{G}}$ is clear, and this implies that the map is proper. $\square$

As the map $L_{\mathcal{G}}$ is surjective, in the following we will denote L by $L_{\mathcal{G}}(V)$. Now consider the inverse limit

$$M = \varprojlim \overline{E_{\mathcal{F}}(V)} = \varprojlim \mathcal{A}_{t_0}(E_{\mathcal{F}}(V)) \cup \partial_{\mathcal{F}} V$$

The space M is compact, as it is an inverse limit of compact spaces, and we will use the map $L_{\mathcal{G}} : V \longrightarrow M$ to define a compactification, as in the previous subsection.

Consider the map

$$i : V \ni x \longrightarrow (x, L_{\mathcal{G}}(x)) \in \hat{V} \times M$$

Let $\overline{V}_{\mathcal{G}}$ be the closure of the image $i(V)$ in $\hat{V} \times M$.

Proposition 84. *The map $i : V \longrightarrow \overline{V}_{\mathcal{G}}$ is a compactification of V. The boundary $\partial_{\mathcal{G}} V = \overline{V}_{\mathcal{G}} \setminus i(V)$ is the set $\varprojlim \partial_{\mathcal{F}} V$.*

Proof: As before, $i(V)$ is homeomorphic to V. The space $\overline{V}_{\mathcal{G}}$ is compact as it is closed in a compact. The complement of $i(V)$ in $\overline{V}_{\mathcal{G}}$ is the set $p_1^{-1}(\infty) \cap \overline{V}_{\mathcal{G}}$, a closed set. Hence the map $i : V \longrightarrow \overline{V}_{\mathcal{G}}$ is a compactification.

Every map $L_{\mathcal{F}}$ is proper, hence the boundary is precisely the set $\lim_{\longleftarrow} \partial_{\mathcal{F}} V$. □

The limit $\partial_{\mathcal{G}} V$ is the spherical quotient of the limit

$$C(\partial_{\mathcal{G}} V) = \varprojlim C(\partial_{\mathcal{F}} V) = \varprojlim \mathcal{A}_0(E_{\mathcal{F}}(V))$$

More explicitly, $C(\partial_{\mathcal{G}} V)$ is a closed subset of the product:

$$\left\{ (x_{\mathcal{F}}) \in \prod_{\mathcal{F} \in P_{\mathcal{G}}} \mathcal{A}_0(E_{\mathcal{F}}(V)) \mid \forall \mathcal{F} \subset \mathcal{F}' : \pi_{\mathcal{F}', \mathcal{F}}(x_{\mathcal{F}'}) = x_{\mathcal{F}} \right\}$$

As before, for every $x \in C(\partial_{\mathcal{G}} V)$, and every $f \in \mathcal{G}$, let $\mathcal{F}$ be a proper finite family containing f. Then the value of the f-coordinate of the point $\pi_{\mathcal{G}, \mathcal{F}}(x)$ does not depend on the choice of the family $\mathcal{F}$. This value will be denoted by x_f. The map

$$C(\partial_{\mathcal{G}} V) \ni x \longrightarrow (x_f)_{f \in \mathcal{G}} \in \mathbb{R}^{\mathcal{G}}$$

identifies $C(\partial_{\mathcal{G}} V)$ with a closed subset of $\mathbb{R}^{\mathcal{G}}$.

5.2 Properties of the boundary

5.2.1 Group actions and compactifications

Let G be a group acting with continuous semi-algebraic maps on a semi-algebraic set $V \subset \mathbb{R}^n$. Suppose that $\mathcal{G}$ is a (possibly infinite) proper family of functions $V \longrightarrow \mathbb{R}_{>0}$, and that $\mathcal{G}$ is invariant for the action of G.

Then the action of G on V extends continuously to an action on the compactification $\overline{V}_{\mathcal{G}}$.

As $\mathcal{G}$ is invariant for the action of G, if we see the limits $L_{\mathcal{G}}(V)$ and $C(\partial_{\mathcal{G}} V)$ as subsets of $\mathbb{R}^{\mathcal{G}}$, then G acts on $L_{\mathcal{G}}(V)$ and $C(\partial_{\mathcal{G}} V)$ by a permutation of the coordinates corresponding to the action on $\mathcal{G}$, and this action induces an action on the spherical quotient of $C(\partial_{\mathcal{G}} V)$, the boundary $\partial_{\mathcal{G}}$.

Note that the map $L_{\mathcal{G}} : V \longrightarrow L_{\mathcal{G}}(V)$ is equivariant for this action, hence the action of G on $\partial_{\mathcal{G}}$ extends continuously the action of G on V.

5.2.2 Injective families and the piecewise linear structure

Let $V \subset \mathbb{R}^n$ be a semi-algebraic set, and let $\mathcal{G}$ be a (possibly infinite) proper family of continuous semi-algebraic functions $V \longrightarrow \mathbb{R}_{>0}$.

An **injective family** is a finite proper subfamily $\mathcal{F} \subset \mathcal{G}$ such that the canonical surjective map $\partial\pi_{\mathcal{F}} : C(\partial_{\mathcal{G}} V) \longrightarrow C(\partial_{\mathcal{F}} V)$ is also injective.

The existence of an injective family is an important tool for us. For $\mathcal{F} \in P_{\mathcal{G}}$, the sets $\mathcal{A}_0(E_{\mathcal{F}}(V)) = C(\partial_{\mathcal{F}} V)$ are all polyhedral complexes, by theorem 53, and the maps

$$\pi_{\mathcal{F}',\mathcal{F}|C(\partial_{\mathcal{F}'}V)} : C(\partial_{\mathcal{F}'} V) \longrightarrow C(\partial_{\mathcal{F}} V)$$

are surjective piecewise linear maps.

Proposition 85. *Suppose that $\mathcal{G}$ contains an injective family. Consider the subset $Q_{\mathcal{G}} \subset P_{\mathcal{G}}$ of all injective families in $\mathcal{G}$. Each of the maps $C(\partial_{\mathcal{G}} V) \longrightarrow C(\partial_{\mathcal{F}} V) \subset \mathbb{R}^{\mathcal{F}}$, for $\mathcal{F} \in Q_{\mathcal{G}}$, is a chart for a piecewise linear structure on $C(\partial_{\mathcal{G}} V)$, and all these charts are compatible, hence they define a canonical piecewise linear structure on $C(\partial_{\mathcal{G}} V)$. As $C(\partial_{\mathcal{G}} V)$ is the cone on $\partial_{\mathcal{G}} V$, a piecewise linear structure is also defined on $\partial_{\mathcal{G}} V$.*

Proof : If $\mathcal{F}, \mathcal{F}' \in Q_{\mathcal{G}}$, then also $\mathcal{F} \cup \mathcal{F}' \in Q_{\mathcal{G}}$, and the maps $C(\partial_{\mathcal{F}' \cup \mathcal{F}} V) \longrightarrow C(\partial_{\mathcal{F}} V)$ and $C(\partial_{\mathcal{F}' \cup \mathcal{F}} V) \longrightarrow C(\partial_{\mathcal{F}'} V)$ are piecewise linear isomorphisms. Hence the charts are compatible. □

For example it is possible to construct a canonical compactification of a complex very affine variety. This construction is closely related to the tropical compactification of [Te]. A **very affine variety** $V \subset (\mathbb{C}^*)^n$ is a closed algebraic subset of the complex torus. The identification $\mathbb{C} = \mathbb{R}^2$ turns V into a real semi-algebraic subset of $\mathbb{R}^{2n}$. We denote by $\mathbb{C}[V]$ the ring of coordinates of V, and by $\mathbb{C}[V]^*$ the group of invertible elements, i.e. the set of polynomials that never vanish on V. For every $f \in \mathbb{C}[V]^*$ we denote by $|f|$ the continuous semi-algebraic function:

$$|f| : V \ni x \longrightarrow |f(x)| \in \mathbb{R}_{>0}$$

We choose a proper family $\mathcal{G}$ in the following way:

$$\mathcal{G} = \{|f| \mid f \in \mathbb{C}[V]^*\}$$

As V is an algebraic subset of the torus, the ring of coordinates is generated by invertible elements, for example the coordinate functions $X_1, \dots, X_n$ and their inverses $X_1^{-1}, \dots, X_n^{-1}$. Then the finite family $\{|X_1|, \dots, |X_n|\} \subset \mathcal{G}$ is a proper family, hence also $\mathcal{G}$ is a proper family. This family defines a compactification

$$i : V \longrightarrow \overline{V}_{\mathcal{G}}$$

Let G be the group of all complex polynomial automorphisms of V. In particular G acts on the semi-algebraic set V with continuous semi-algebraic maps. This action preserves $\mathbb{C}[V]^*$, hence it also preserves the family $\mathcal{G}$. Then the action of G on V extends to an action on the compactification $\overline{V}_{\mathcal{G}}$.

The family $\mathcal{G}$ is infinite (and uncountable), yet it is possible to find an injective family. To do this we use the same technique used in [Te] to construct the intrinsic torus. By [ST, Rem. 2.10], the group $\mathbb{C}[V]^*/\mathbb{C}^*$ is finitely generated.

Proposition 86. *Let $f_1, \dots, f_m \in \mathbb{C}[V]^*$ be representatives of generators of the group $\mathbb{C}[V]^*/\mathbb{C}^*$. Then the family $\mathcal{F} = \{|f_1|, \dots, |f_m|\} \subset \mathcal{G}$ is an injective family.*

Proof : Suppose, by contradiction, that the projection $\pi_{\mathcal{G},\mathcal{F}|C(\partial_{\mathcal{G}} V)} : C(\partial_{\mathcal{G}} V) \longrightarrow C(\partial_{\mathcal{F}} V)$ is not injective. Then there exists an $x \in C(\partial_{\mathcal{F}} V)$ such that $\pi^{-1}_{\mathcal{G},\mathcal{F}|C(\partial_{\mathcal{G}} V)}(x)$ contains at least two elements $y, y' \in C(\partial_{\mathcal{G}} V)$. As $y \neq y'$ there exists an element $f \in \mathbb{C}[V]^*$ such that the coordinates $y_{|f|}$ and $y'_{|f|}$ differ.

This means that also the projection

$$\pi_{\mathcal{F}\cup\{|f|\},\mathcal{F}|C(\partial_{\mathcal{F}\cup\{|f|\}} V)} : C(\partial_{\mathcal{F}\cup\{|f|\}} V) \longrightarrow C(\partial_{\mathcal{F}} V)$$

is not injective. The set $C(\partial_{\mathcal{F}\cup\{|f|\}} V)$ is simply the logarithmic limit set $\mathcal{A}_0(E_{\mathcal{F}\cup\{|f|\}}(V)) \subset \mathbb{R}^{m+1}$, and, by corollary 69 it is the image, under the componentwise valuation map, of the set $\overline{E_{\mathcal{F}\cup\{|f|\}}(V)} \subset H(\overline{\mathbb{R}}^{\mathbb{R}})$.

By the hypothesis on $f_1, \dots, f_m$, there exists integers $e_i \in \mathbb{Z}$ and a number $c \in \mathbb{C}$ such that $f = c \prod_{i=1}^m f_i^{e_i}$. Hence $|f| = |c| \prod_{i=1}^m |f_i|^{e_i}$, and for every $z \in H(\overline{\mathbb{R}}^{\mathbb{R}})$, $v(|f(z)|) = \sum_{i=1}^m e_i v(|f_i(z)|)$. The valuation of $|f(z)|$ is determined by the valuations of $|f_i(z)|$, hence the map $\pi_{\mathcal{F}\cup\{|f|\},\mathcal{F}|C(\partial_{\mathcal{F}\cup\{|f|\}} V)}$ is injective, a contradiction. $\square$

Corollary 87. *In particular the boundary $\partial_{\mathcal{G}} V$ of a very affine variety has a natural piecewise linear structure, and the group G acts on the compactification $\overline{V}_{\mathcal{G}}$ with an action by complex polynomial maps on the interior part, and by piecewise linear maps on the boundary.* $\square$

Unfortunately, when working with real algebraic sets, a general technique of this kind for constructing injective families does not work. If V is a real algebraic set, a polynomial function that never vanish on V is not necessarily invertible in the ring of coordinates. It is invertible in the ring of regular functions, but this ring is not always a finitely generated $\mathbb{R}$-algebra, hence the group of invertible elements is not finitely generated.

The following proposition gives a sufficient hypothesis for the existence of injective families. This proposition can be used to prove the existence of injective families for the compactification of the Teichmüller space of the once punctured torus. For the Teichmüller space of a general surface we will need to use a bit more of Teichmüller theory.

Proposition 88. *Let $V \subset \mathbb{R}^n$ be a real semi-algebraic set, and let $\mathcal{G}$ be a proper family of positive continuous semi-algebraic functions on V. Suppose that there exists a proper family $\mathcal{F} = \{f_1, \dots, f_m\} \subset \mathcal{G}$ such that for every element $f \in \mathcal{G}$ there exists a Laurent polynomial $P(x_1, \dots, x_m)$ with real and positive coefficients such that $f = P(f_1, \dots, f_m)$. Then $\mathcal{F}$ is an injective family.*

Proof : As before, suppose, by contradiction, that the projection $\pi_{\mathcal{G},\mathcal{F}|C(\partial_{\mathcal{G}}V)} : C(\partial_{\mathcal{G}}V) \longrightarrow C(\partial_{\mathcal{F}}V)$ is not injective. Then there exists an $x \in C(\partial_{\mathcal{F}}V)$ such that $\pi^{-1}_{\mathcal{G},\mathcal{F}|C(\partial_{\mathcal{G}}V)}(x)$ contains at least two elements $y, y' \in C(\partial_{\mathcal{G}}V)$. As $y \neq y'$ there exists an element $f \in \mathcal{G}$ such that the coordinates y_f and y'_f differ.

This means that also the projection

$$\pi_{\mathcal{F}\cup\{f\},\mathcal{F}|C(\partial_{\mathcal{F}\cup\{f\}}V)} : C(\partial_{\mathcal{F}\cup\{f\}}V) \longrightarrow C(\partial_{\mathcal{F}}V)$$

is not injective. The set $C(\partial_{\mathcal{F}\cup\{f\}}V)$ is simply the logarithmic limit set $\mathcal{A}_0(E_{\mathcal{F}\cup\{f\}}(V)) \subset \mathbb{R}^{m+1}$, and, by corollary 69 it is the image, under the componentwise valuation map, of the set $\overline{E_{\mathcal{F}\cup\{f\}}(V)} \subset H(\overline{\mathbb{R}}^{\mathbb{R}})$.

By hypothesis we have $f = P(f_1, \dots, f_m)$, where the coefficients of P are real and positive, hence for every $z \in H(\overline{\mathbb{R}}^{\mathbb{R}})$, the valuation of $f(z)$ is determined by the valuations of $f_i(z)$, hence the map $\pi_{\mathcal{F}\cup\{|f|\},\mathcal{F}|C(\partial_{\mathcal{F}\cup\{|f|\}}V)}$ is injective, a contradiction. □

Note that in the previous proposition we had to require the Laurent polynomial $P(x_1, \dots, x_n)$ to have positive coefficients. With the weaker hypothesis that the polynomial P is positive whenever the variables $x_1, \dots, x_n$ are positive, the statement becomes false. For example, let $V = (\mathbb{R}_{>0})^2$. The family of functions $\mathcal{F} = \{x, y\}$ is proper, and $C(\partial_{\mathcal{F}}V)$ is simply the plane $\mathbb{R}^2$. Consider the family $\mathcal{G} = \{x, y, x^2 + (y-1)^2\}$. The set $E_{\mathcal{G}}(V) \subset \mathbb{R}^3$ and the logarithmic limit set $C(\partial_{\mathcal{G}}V)$ are represented in figure 5.1. The map $\pi_{\mathcal{G},\mathcal{F}}$ is not injective, hence $\mathcal{F}$ is not an injective family of $\mathcal{G}$.

5.2.3 Non-archimedean description

Let $V \subset \mathbb{R}^n$ be a semi-algebraic set, and let $\mathcal{G}$ be a (possibly infinite) proper family of continuous semi-algebraic functions $V \longrightarrow \mathbb{R}_{>0}$.

Let $\mathbb{F}$ be a real closed non-archimedean field with finite rank extending $\mathbb{R}$. The convex hull of $\mathbb{R}$ in $\mathbb{F}$ is a valuation ring denoted by $\mathcal{O}_{\leq}$. This valuation ring defines a valuation $v : \mathbb{F}^* \longrightarrow \Lambda$, where Λ is an ordered abelian group. As $\mathbb{F}$ has finite rank, the group Λ has only finitely many convex subgroups $0 = \Lambda_0 \subset \Lambda_1 \subset \cdots \subset \Lambda_r = \Lambda$. The number r of convex subgroups is the **rank** of the field $\mathbb{F}$.

The quotient $\Lambda \longrightarrow \Lambda/\Lambda_{r-1}$ is an ordered group of rank one, hence it is isomorphic to a subgroup of $\mathbb{R}$. We fix one of these isomorphisms, and

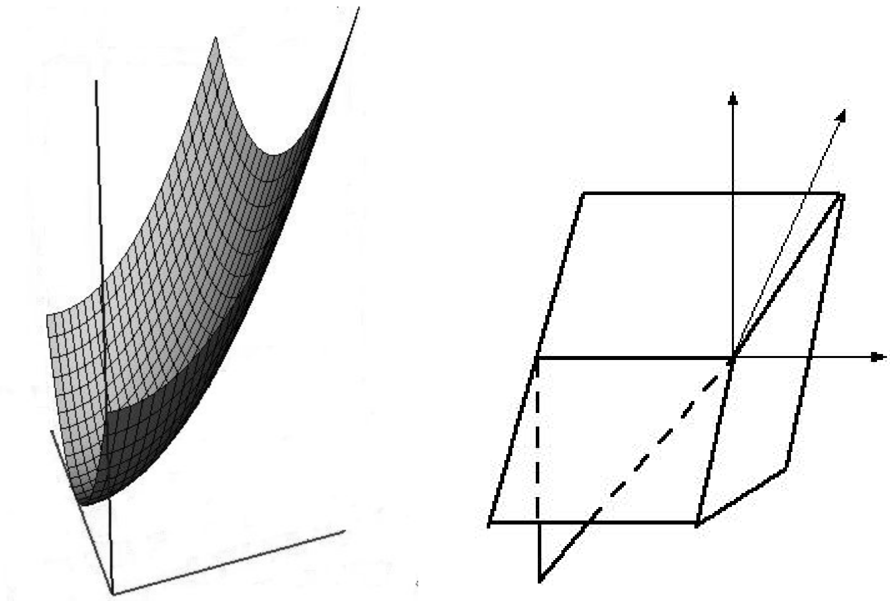

Figure 5.1: $V = \{(x,y,z) \in (\mathbb{R}_{>0})^3 \mid z = x^2 + (y-1)^2\}$ (left picture), then $\mathcal{A}_0(V)$ is made up of four faces (right picture).

we denote by $\overline{v}$ the composition of the valuation v with the quotient map $\Lambda \longrightarrow \Lambda/\Lambda_{r-1}$, another valuation of $\mathbb{F}$ that is real valued:

$$\overline{v} : \mathbb{F}^* \longrightarrow \mathbb{R}$$

This valuation induces a norm and a log map:

$$\mathbb{F} \ni h \longrightarrow \;\parallel h \parallel = \exp(-\overline{v}(h)) \in \mathbb{R}_{\geq 0}$$

$$\mathrm{Log} : (\mathbb{F}_{>0})^n \ni (h_1, \dots, h_n) \longrightarrow (\log(\parallel h_1 \parallel), \dots, \log(\parallel h_n \parallel)) \in \mathbb{R}^n$$

Let $\overline{V}$ be the extension of V to $\mathbb{F}$, a semi-algebraic subset of $(\mathbb{F}_{>0})^n$. Let $\overline{\mathcal{G}} = \{\overline{f} \mid f \in \mathcal{G}\}$, where $\overline{f} : \overline{V} \longrightarrow \mathbb{F}_{>0}$ is the extension of the function $f : V \longrightarrow \mathbb{R}_{>0}$.

Let $\mathcal{F} = \{f_1, \dots, f_m\} \subset \mathcal{G}$ be a finite proper family, and let $\overline{\mathcal{F}} = \{\overline{f_1}, \dots, \overline{f_m}\} \subset \overline{\mathcal{G}}$ be the corresponding family of extensions. We will denote by $\overline{E_{\mathcal{F}}} : \overline{V} \longrightarrow (\mathbb{F}_{>0})^m$ the extension of the map $E_{\mathcal{F}}$.

Proposition 89. *The image of the map*

$$\mathrm{Log} : (\mathbb{F}_{>0})^n \supset \overline{E_{\mathcal{F}}}(\overline{V}) \ni x \longrightarrow (-\overline{v}(x_1), \dots, -\overline{v}(x_n)) \in \mathbb{R}^m$$

is contained in the logarithmic limit set $\mathcal{A}_0(E_{\mathcal{F}}(V))$.

Proof : Let $t \in \mathbb{F}$ be an element such that $t > 0$ and $\overline{v}(t) = 1$. Consider the subfield $\mathbb{R}(t) \subset \mathbb{F}$. The order induced by $\mathbb{F}$ has the property that $t > 0$

and $\forall x \in \mathbb{R}_{>0} : t < x$. Hence $\mathbb{F}$ contains the real closure of $\mathbb{R}(t)$ with reference to this order, i.e. $H(\overline{\mathbb{R}})$. Moreover the valuation $\overline{v}$ on $\mathbb{F}$ restricts to the valuation we have defined on $H(\overline{\mathbb{R}})$, as, if $\mathcal{O}_{\leq}$ is the valuation ring of $\mathbb{F}$, $\mathcal{O}_{\leq} \cap H(\overline{\mathbb{R}})$ is precisely the valuation ring $\mathcal{O}$ of $H(\overline{\mathbb{R}})$.

We recall that, by corollary 70, $\mathrm{Log}(\overline{V} \cap H(\overline{\mathbb{R}})) \subset \mathcal{A}_0(E_{\mathcal{F}}(V))$. By reporting, word by word, the proof of theorem 67, we can prove that the image $\mathrm{Log}(\overline{V})$ is contained in the closure of $\mathrm{Log}(\overline{V} \cap H(\overline{\mathbb{R}}))$, i.e. it is contained in $\mathcal{A}_0(E_{\mathcal{F}}(V))$. □

In other words, the image of the map

$$\mathrm{Log}_{\mathcal{F}} = \mathrm{Log} \circ \overline{E_{\mathcal{F}}} : \overline{V} \ni x \longrightarrow (-\overline{v}(f_1(x)), \ldots, -v(f_m(x))) \in \mathbb{R}^m$$

is contained in $\mathcal{A}_0(E_{\mathcal{F}}(V)) = C(\partial_{\mathcal{F}} V)$.

The system of maps $\mathrm{Log}_{\mathcal{F}} : \overline{V} \longrightarrow C(\partial_{\mathcal{F}} V)$, defined for every $\mathcal{F} \in P_{\mathcal{G}}$, induces by the universal property a well defined map $\mathrm{Log}_{\mathcal{G}} : V \longrightarrow C(\partial_{\mathcal{G}} V)$. The map $\mathrm{Log}_{\mathcal{G}}$ can be identified with the map

$$\overline{V} \ni x \longrightarrow (-\overline{v}(f(x)))_{f \in \mathcal{G}} \in C(\partial_{\mathcal{G}} V) \subset \mathbb{R}^{\mathcal{G}}$$

This map is not surjective for every field $\mathbb{F}$. The aim of this subsection is to show that there exists a real closed non-archimedean field of finite rank $\mathbb{F}$ such that the map $\mathrm{Log}_{\mathcal{G}}$ is surjective. As a consequence, the boundary $\partial_{\mathcal{G}} V$ is the spherical quotient of the set $\mathrm{Log}_{\mathcal{G}}(\overline{V})$.

The easiest case is when $\mathcal{G}$ has an injective family. In this case we use the Hardy field $H(\overline{\mathbb{R}}^{\mathbb{R}})$, as in subsection 4.1.1.

Proposition 90. *Suppose that the family $\mathcal{G}$ contains an injective family $\mathcal{F}$. Then if $\mathbb{F} = H(\overline{\mathbb{R}}^{\mathbb{R}})$, the image of the map $\mathrm{Log}_{\mathcal{G}}$ is the whole $C(\partial_{\mathcal{G}} V)$.*

Proof : Let $y \in C(\partial_{\mathcal{G}} V)$. By corollary 69, the map $\mathrm{Log}_{\mathcal{F}} : \overline{V} \longrightarrow \mathcal{A}_0(\overline{E_{\mathcal{F}}}(V))$ is surjective, hence there exists $x \in \overline{V}$ such that $\mathrm{Log}_{\mathcal{F}}(x) = \pi_{\mathcal{G},\mathcal{F}}(y)$. Hence $\pi_{\mathcal{G},\mathcal{F}}(\mathrm{Log}_{\mathcal{G}}(x)) = \pi_{\mathcal{G},\mathcal{F}}(y)$, and, as $\mathcal{F}$ is an injective family, $\mathrm{Log}_{\mathcal{G}}(x) = y$. □

We can get a similar result even if we remove the hypothesis of the existence of an injective family. To do this we will need to use a larger field. If V is a semi-algebraic set of dimension r, consider the group $\mathbb{R}^r$, with the lexicographic order. We will use the field of transfinite Puiseaux series with real coefficients and exponents in $\mathbb{R}^r$:

$$\mathbb{R}((t^{\mathbb{R}^r})) = \left\{ \sum_{r \in \mathbb{R}^r} a_r t^r \mid a_r \in \mathbb{R}, \{r \in \mathbb{R}^r \mid a_r \neq 0\} \text{ is well ordered} \right\}$$

This is a real closed non-archimedean field of rank r, with a surjective valuation $v : \mathbb{R}((t^{\mathbb{R}}))^* \longrightarrow \mathbb{R}^r$, and a quotient surjective valuation $\overline{v} : \mathbb{R}((t^{\mathbb{R}}))^* \longrightarrow \mathbb{R}$.

The rest of this subsection is dedicated to the proof of theorem 96: if $\mathbb{F} = \mathbb{R}((t^{\mathbb{R}^r}))$, the map $\mathrm{Log}_{\mathcal{G}}$ is surjective.

Let $y \in C(\partial_{\mathcal{G}} V)$, and let $[y] \in \partial_{\mathcal{G}} V$ be the corresponding point. The set of all semi-algebraic subsets of V is a Boolean algebra. Consider the set ϕ_y of all semi-algebraic subsets $A \subset V$ such that there exists a neighborhood U of $[y]$ in $\overline{V}_{\mathcal{G}}$ with $U \cap V \subset A$. The set ϕ_y is a **filter** in the Boolean algebra of all semi-algebraic subsets of V, i.e. ϕ_y is closed for finite intersections, if $A \in \phi_y$ and $A \subset B$, then $B \in \phi_y$, $V \in \phi_y$, $\emptyset \notin \phi_y$. The ultrafilter lemma states that every filter is contained in an **ultrafilter**, i.e. a filter such that for every semi-algebraic set $A \subset V$, A or $V \setminus A$ is in the ultrafilter. We choose an ultrafilter α containing ϕ_y. Note that if $A \in \alpha$, then the complement of A in $\overline{V}_{\mathcal{G}}$ does not contain a neighborhood of $[y]$, hence the closure of A in $\overline{V}_{\mathcal{G}}$ contains the boundary point $[y]$. Intuitively speaking, the choice of an ultrafilter α containing ϕ_y can be interpreted as the choice of a way for converging to $[y]$.

Now, given an ultrafilter α as above, we construct a field $\mathfrak{K}(\alpha)$, as in subsection 4.1.1. See also [Co, sez. 5.3] for a more detailed reference. Consider the ring $S(V)$ of all semi-algebraic functions from V to $\mathbb{R}$. Let $I(\alpha)$ be the subset of all functions whose zero locus is in α. We define $\mathfrak{K}(\alpha) = S(V)/I(\alpha)$. It is easy to show that every non-zero element is invertible, hence $\mathfrak{K}(\alpha)$ is a field. Equivalently, $\mathfrak{K}(\alpha)$ can be defined as the quotient of $S(V)$ under the relation $f \sim g$ if and only if $\{f(x) = g(x)\} \in \alpha$. This second definition is the one used in subsection 4.1.1 and in [Co, sez. 5.3]. We denote the equivalence class of a semi-algebraic function f by $[f]$. In the same way we can define the order on $\mathfrak{K}(\alpha)$: $[f] \leq [g]$ if and only if $\{f(x) \leq g(x)\} \in \alpha$, and this definition does not depend on the choice of the representative. Hence $\mathfrak{K}(\alpha)$ is an ordered field, and has an $\mathcal{OS}$-structure.

Proposition 91. *Given an $(L_{\mathcal{OS}})$-formula $\phi(x_1, \ldots, x_n)$, and given definable functions $f_1, \ldots, f_n$, we have:*

$$\mathfrak{K}(\alpha) \vDash \phi([f_1], \ldots, [f_n]) \Leftrightarrow \exists S \in \alpha : \forall t \in S : \overline{\mathbb{R}} \vDash \phi(f_1(t), \ldots, f_n(t))$$

Proof: It is identical to the proof of theorem 61. See also [Co, thm. 5.8]. $\square$

In particular $\mathfrak{K}(\alpha)$ is a real closed field. For every element $a \in \mathbb{R}$, the constant function with value a defines an element of $\mathfrak{K}(\alpha)$ that is identified with a. This defines an an embedding $\mathbb{R} \longrightarrow \mathfrak{K}(\alpha)$.

Consider an embedding $V \longrightarrow \mathbb{R}^n$. For every semi-algebraic subset $A \subset V$, we denote by $\overline{A}$ its Zariski-closure in $\mathbb{R}^n$. Consider the set

$$W_\alpha = \bigcap_{A \in \alpha} \overline{A}$$

Every infinite intersection of Zariski-closed sets can be written as a finite intersection, hence W_α is a Zariski-closed subset of $\mathbb{R}^n$. Moreover W_α is irreducible and $W_\alpha \cap V \in \alpha$. We will call the set W_α the **algebraic support** of the ultrafilter α. Note that the algebraic support depends on the chosen embedding $V \longrightarrow \mathbb{R}^n$.

Every polynomial $f \in \mathbb{R}[x_1, \dots, x_n]$ defines a semi-algebraic function $f : V \longrightarrow \mathbb{R}$, and an element $[f] \in \mathfrak{K}(\alpha)$. Two polynomials define the same element of $\mathfrak{K}(\alpha)$ if and only if the coincide on W_α. In other words, if $\mathbb{R}[W_\alpha]$ and $\mathbb{R}(W_\alpha)$ are, respectively, the coordinate ring of W_α and its field of fractions, there is an embedding $\mathbb{R}(W_\alpha) \subset \mathfrak{K}(\alpha)$. As every semi-algebraic function satisfies a polynomial equation, the field $\mathfrak{K}(\alpha)$ is algebraic over $\mathbb{R}(W_\alpha)$, it is its real closure with respect to the ordering induced by the embedding.

The convex hull of $\mathbb{R}$ in $\mathfrak{K}(\alpha)$ is a valuation ring denoted by $\mathcal{O}_\leq$. As above, this valuation ring defines a valuation $v : \mathfrak{K}(\alpha)^* \longrightarrow \Lambda$, where Λ is an ordered abelian group. If V has dimension r, the transcendence degree over $\mathbb{R}$ of the field $\mathbb{R}(W_\alpha)$ is at most r, hence, by [ZS2, cap. VI, thm. 3, cor. 1] and [ZS2, cap. VI, thm. 15], the group Λ has rank at most r.

As before the composition of the valuation v with the quotient map $\Lambda \longrightarrow \Lambda/\Lambda_{r-1}$, defines another valuation of $\mathfrak{K}(\alpha)$ that is real valued:

$$\overline{v} : \mathfrak{K}(\alpha)^* \longrightarrow \mathbb{R}$$

Proposition 92. *Let V be a semi-algebraic set, and let $\mathcal{F}$ be a finite proper family of continuous positive semi-algebraic functions on V. If g is a continuous positive semi-algebraic function, then there exist $A, B \in \mathbb{R}_{>0}$ and $n \in \mathbb{N}$ such that:*

$$\forall x \in V : g(x) \leq A \left(\sum_{f \in \mathcal{F}} f(x) + f^{-1}(x) \right)^n + B$$

Proof : Suppose, by contradiction, that the statement is false. Fix two increasing unbounded sequences $A_n, B_n \in \mathbb{R}$, then for all $n \in \mathbb{N}$ there exists a point $x_n \in V$ such that

$$g(x_n) > A_n \left(\sum_{f \in \mathcal{F}} f(x_n) + f^{-1}(x_n) \right)^n + B_n$$

There is no subsequence of x_n contained in a compact subset of V, because $g(x_n)$ is unbounded. If we consider the compactification $\overline{V}_{\mathcal{F} \cup \{g\}}$, then we can extract a subsequence x_{n_k} converging to a point $[z] \in \partial_{\mathcal{F} \cup \{g\}} V$. As g grows faster than every $f \in \mathcal{F}$ along the sequence x_{k_n}, a point $z \in C(\partial_{\mathcal{F} \cup \{g\}} V)$ corresponding to $[z]$ has coordinates $z_g \neq 0$ and $z_f = 0$ for all $f \in \mathcal{F}$.

The logarithmic limit set $\mathcal{A}_0(E_{\mathcal{F} \cup \{g\}}(V))$ contains the point $(0, 0, \dots, 1)$, where the 1 is in the coordinate corresponding to g. Then, by theorem 47 there exists a sequence $y_k \in V$ such that $g(y_k) \to \infty$ and $f(y_k)$ is bounded for all $f \in \mathcal{F}$. But this is absurd because $\mathcal{F}$ is a proper family. $\square$

Corollary 93. *For every $f \in \mathcal{G}$ such that $y_f \neq 0$, we have $\overline{v}([f]) \neq 0$.*

Proof : It follows from the previous proposition. □

Proposition 94. *Let $f, g \in \mathcal{G}$, and suppose that $y_f \neq 0$. Then*

$$\frac{-\overline{v}([f])}{-\overline{v}([g])} = \frac{y_f}{y_g}$$

Proof : For every $\varepsilon > 0$ there exists a neighborhood U of $[y]$ in $\overline{V}_{\mathcal{G}}$ such that

$$\forall x \in U : \left| \frac{\log_e(f(x))}{\log_e(g(x))} - \frac{y_f}{y_g} \right| < \varepsilon$$

$$\forall x \in U : g(x)^{\left(\frac{y_f}{y_g} - \varepsilon\right)} < f(x) < g(x)^{\left(\frac{y_f}{y_g} + \varepsilon\right)}$$

Hence, for every rational number $r < \frac{y_f}{y_g}$ there exists a neighborhood U of $[y]$ in $\overline{V}_{\mathcal{G}}$ such that

$$\forall x \in U : g(x)^r < f(x)$$

and for every rational number $r > \frac{y_f}{y_g}$ there exists a neighborhood U of $[y]$ in $\overline{V}_{\mathcal{G}}$ such that

$$\forall x \in U : f(x) < g(x)^r$$

As all these neighborhoods U are in α, and the rational powers are semi-algebraic functions, these inequalities hold on the field $\mathfrak{K}(\alpha)$: for every rational number $r < \frac{y_f}{y_g}$ we have $[g]^r < [f]$ and for every rational number $r > \frac{y_f}{y_g}$ we have $[f] < [g]^r$. Hence

$$\frac{-\overline{v}([f])}{-\overline{v}(g)} = \frac{y_f}{y_g}$$

□

Now let $\overline{V} \subset (\mathfrak{K}(\alpha))^n$ be the extension of $V \subset \mathbb{R}^n$ to the field $\mathfrak{K}(\alpha)$. For every $f \in \mathcal{G}$, let $\overline{f}$ be the extension of f to the field $\mathfrak{K}(\alpha)$.

Proposition 95. *There exists a point $x \in \overline{V}$ such that*

$$(-v(\overline{f}(x)))_{f \in \mathcal{G}} = y$$

Proof: If $x_1, \dots, x_n$ are the restriction to V of the coordinate functions of $\mathbb{R}^n$, then the point $([x_1], \dots, [x_n]) \in (\mathfrak{K}(\alpha))^n$ is in $\overline{V}$, because the coordinate functions restricted to V satisfy the first order formulas satisfied by the points of V. Note also that $\overline{f}([x_1], \dots, [x_n]) = [f]$, hence, by the previous proposition, the point $([x_1], \dots, [x_n])$ is what we are searching for. □

Theorem 96. *Let $V \subset \mathbb{R}^n$ be a semi-algebraic set of dimension r, and let $\mathcal{G}$ be a proper family of positive continuous semi-algebraic functions on V. If $\mathbb{F} = \mathbb{R}((t^{\mathbb{R}^r}))$, and $\overline{V}$ is the extension of V to the field $\mathbb{F}$, then $\mathrm{Log}_{\mathcal{G}}(\overline{V}) = C(\partial_{\mathcal{G}} V)$.*

Proof: By the previous proposition, if $y \in C(\partial_{\mathcal{G}} V)$, and α is an ultrafilter associated to y, there exists a real closed field $\mathfrak{K}(\alpha)$, with rank at most r, such that the extension of V to $\mathfrak{K}(\alpha)$ has a point $x \in (\mathfrak{K}(\alpha))^n$ such that $\mathrm{Log}_{\mathcal{G}}(x) = y$.

The conclusion follows from the fact that every real closed non-archimedean field of rank at most r can be embedded in $\mathbb{R}((t^{\mathbb{R}^r}))$ (see [Gl37]). Hence the image of the point x in $(\mathbb{R}((t^{\mathbb{R}^r}))^n$ is in $\overline{V}$ and $\mathrm{Log}_{\mathcal{G}}(x) = y$. $\square$

5.3 Spaces of geometric structures

5.3.1 Teichmüller spaces

Let $\overline{S} = \Sigma_g^k$ be an orientable compact surface of genus g, with $k \geq 0$ boundary components and with $\chi(\overline{S}) < 0$, and let S be its interior. We want to construct a compactification of the Teichmüller space $\mathcal{T}^{cf}_{\mathbb{H}^2}(S)$, using the structure of semi-algebraic set it inherits from its identification with a closed semi-algebraic subset of $\overline{\mathrm{Char}}(\pi_1(S), SL_2(\mathbb{R}))$ and $\mathrm{Char}(\pi_1(S), SL_2(\mathbb{R}))$.

Let $\mathcal{G} = \{J_\gamma\}_{\gamma \in \pi_1(S)}$, the positive trace functions, as defined in subsection 2.2.2. As we said in subsection 1.4.3, there exists a finite subset $A \subset \pi_1(S)$ such that the family of functions $\mathcal{F}_A = \{J_\gamma\}_{\gamma \in A}$ generates the ring of coordinates of $\mathrm{Char}(\pi_1(S), SL_2(\mathbb{R}))$. As $\mathcal{T}^{cf}_{\mathbb{H}^2}(S)$ is closed in $\mathrm{Char}(\pi_1(S), SL_2(\mathbb{R}))$, the family $\mathcal{F}_A$ is proper, hence the family $\mathcal{G}$ is too.

The family $\mathcal{G}$ defines a compactification

$$\overline{\mathcal{T}^{cf}_{\mathbb{H}^2}(S)}_{\mathcal{G}} = \mathcal{T}^{cf}_{\mathbb{H}^2}(S) \cup \partial_{\mathcal{G}} \mathcal{T}^{cf}_{\mathbb{H}^2}(S)$$

As the family $\mathcal{G}$ is invariant for the action of the mapping class group of S, the action of the mapping class group extends continuously to an action on $\overline{\mathcal{T}^{c}_{\mathbb{H}^2}(S)}_{\mathcal{G}}$.

We want to prove that this compactification of the Teichmüller space $\mathcal{T}^{cf}_{\mathbb{H}^2}(S)$ is the same as the one constructed by Thurston, see [FLP] for details on Thurston's work. The boundary constructed by Thurston, here denoted by $\overline{\mathcal{T}^{cf}_{\mathbb{H}^2}(S)}_T = \mathcal{T}^{cf}_{\mathbb{H}^2}(S) \cup \partial_T \mathcal{T}^{cf}_{\mathbb{H}^2}(S)$ can be described by using projectivized length spectra of measured laminations, as in [FLP]. If l is a measured lamination, for every element $\gamma \in \pi_1(S)$, the measure of γ, denoted by $I(l, \gamma)$, is the infimum of the measures of all the closed curves in S that are freely homotopic to γ. The cone over the boundary, $C(\partial_T \mathcal{T}^{cf}_{\mathbb{H}^2}(S))$, can be identified with the subset of $\mathbb{R}^{\pi_1(S)}$ consisting of points of the form $(I(l,\gamma))_{\gamma \in \pi_1(S)}$. A sequence $(x_n) \subset \mathcal{T}^{cf}_{\mathbb{H}^2}(S)$ converges to a point $x_0 \in \partial_T \mathcal{T}^{cf}_{\mathbb{H}^2}(S)$ if and only if the projectivized length spectra $[\ell_\gamma(x_n)]_{\gamma \in \pi_1(S)}$ of the hyperbolic structures converges to the projectivized length spectra of the measured lamination corresponding to x_0.

Proposition 97. *The compactification* $\overline{\mathcal{T}^{cf}_{\mathbb{H}^2}(S)}_{\mathcal{G}}$ *is isomorphic to the compactification* $\overline{\mathcal{T}^{cf}_{\mathbb{H}^2}(S)}_{T}$ *constructed by Thurston. More precisely, the sets* $C(\partial_T \mathcal{T}^{cf}_{\mathbb{H}^2}(S))$ *and* $C(\partial_T \mathcal{T}^{cf}_{\mathbb{H}^2}(S))$ *coincide when identified with subsets of* $\mathbb{R}^{\pi_1(S)}$*, and the notion of convergence of sequences on* $\mathcal{T}^{cf}_{\mathbb{H}^2}(S)$ *to the boundary points also coincide.*

Proof : If $(x_n) \subset \mathcal{T}^{cf}_{\mathbb{H}^2}(S)$ is a sequence converging to l in Thurston compactification, then $I(l,\gamma)_{\gamma\in\pi_1(S)}$ is the limit of the sequence $[\ell_\gamma(x_n)]_{\gamma\in\pi_1(S)}$, and, by the relation $J_\gamma([h]) = 2\cosh(\frac{1}{2}\ell_c([h]))$ in subsection 2.2.2, this is equal to the limit of the sequence $[\log_e(J_\gamma(x_n))]_{\gamma\in\pi_1(S)}$. □

5.3.2 Spaces of convex projective structures on manifolds

Let M be a closed n-manifold such that the fundamental group $\pi_1(M)$ has trivial virtual center, it is Gromov hyperbolic, and it is torsion free (note that every closed hyperbolic n-manifold whose fundamental group is torsion-free satisfies the hypotheses). We want to construct a compactification of the space $\mathcal{T}^c_{\mathbb{RP}^n}(M)$ of marked convex projective structures on M, using the structure of semi-algebraic set it inherits from its identification with a connected component of $\overline{\text{Char}}(\pi_1(M), SL_n(\mathbb{R}))$.

For every element $p \in \mathcal{T}^c_{\mathbb{RP}^n}(M)$ and $\gamma \in \pi_1(M)$, we denote by $\ell_\gamma(p)$ the translation length of γ, see subsection 2.3.3, and we denote by $e_\gamma(p)$ the ratio $\frac{\lambda_1}{\lambda_{n+1}}$ between the eigenvalues of maximum and minimum modulus of the conjugacy class of matrices $p(\gamma)$. Then the function

$$e_\gamma : \mathcal{T}^c_{\mathbb{RP}^n}(M) \longrightarrow \mathbb{R}_{>0}$$

is a semi-algebraic function on $\mathcal{T}^c_{\mathbb{RP}^n}(M)$, such that $\log_e(e_\gamma(p)) = \ell_\gamma(p)$.

Let $\mathcal{G} = \{e_\gamma\}_{\gamma\in\pi_1(M)}$. As we said in subsection 1.4.3, there exists a finite subset $A \subset \pi_1(M)$ such that the family of functions $\{I_\gamma\}_{\gamma\in A}$ generates the ring of coordinates of $\text{Char}(\pi_1(M), SL_{n+1}(\mathbb{R}))$.

Lemma 98. *Let* $p \in \mathcal{T}^c_{\mathbb{RP}^n}(M)$ *and* $\gamma \in \pi_1(M)$*. Denote by* $\lambda_1, \ldots, \lambda_{n+1}$ *the complex eigenvalues of* $p(\gamma)$*, with non-increasing absolute values. Then:*

$$|\operatorname{tr}(p(\gamma))| \leq (n+1)\frac{\lambda_1}{\lambda_{n+1}}$$

Proof : First we recall that λ_1 and λ_{n+1} are real and positive, hence the statement makes sense. As $|\lambda_1| \geq |\lambda_i|$, then $|\operatorname{tr}(p(\gamma))| \leq (n+1)\lambda_1$. As $p(\gamma) \in SL_{n+1}(\mathbb{R})$, then $\lambda_{n+1} \leq 1$. □

Proposition 99. *The family* $\mathcal{F}_A = \{e_\gamma\}_{\gamma\in A}$ *is proper.*

Proof: Suppose that $(x_n) \subset \mathcal{T}^c_{\mathbb{RP}^n}(M)$ is a sequence that is not contained in any compact subset of $\mathcal{T}^c_{\mathbb{RP}^n}(M)$. Suppose, by contradiction, that the image $E_{\mathcal{F}_A}(x_n) \subset (\mathbb{R}_{>0})^{\mathcal{F}_A}$ is bounded. Then, by previous lemma, the functions $\{I_\gamma\}_{\gamma\in A}$ are bounded on (x_n), hence the sequence (x_n) converges (up to subsequences) to a point of $\mathrm{Char}(\pi_1(M), SL_{n+1}(\mathbb{R}))$. As $\mathcal{T}^c_{\mathbb{RP}^n}(M)$ is closed in $\mathrm{Char}(\pi_1(M), SL_{n+1}(\mathbb{R}))$, we have a contradiction. $\square$

We have proved that the family $\mathcal{G}$ is a proper family, hence it defines a compactification

$$\overline{\mathcal{T}^c_{\mathbb{RP}^n}(M)}_{\mathcal{G}} = \mathcal{T}^c_{\mathbb{RP}^n}(M) \cup \partial_{\mathcal{G}}\mathcal{T}^c_{\mathbb{RP}^n}(M)$$

As the family $\mathcal{G}$ is invariant for the action of the mapping class group of M, the action of the mapping class group extends continuously to an action on $\overline{\mathcal{T}^c_{\mathbb{RP}^n}(M)}_{\mathcal{G}}$.

Note that this compactification is constructed taking the limits of the functions $\log_e \circ e_\gamma$, i.e. the limits of the translation length functions ℓ_γ.

5.4 Piecewise linear structure on the boundary of Teichmüller spaces

5.4.1 Existence of an injective family

To show that our construction of the boundary defines a piecewise linear structure on it, we only need to find an injective family. First we show how to use proposition 88 to find an injective family for the Teichmüller space of the once punctured torus. Then we pass to the general case.

Proposition 100. *Let $A \subset \pi_1(S)$ be a finite subset such that the family of functions $\{J_\gamma\}_{\gamma\in A}$ generates the ring of coordinates of* $\mathrm{Char}(\pi_1(S), SL_2(\mathbb{R}))$*. Then the family $\mathcal{F}_A = \{J_\gamma\}_{\gamma\in A}$ is proper.*

Proof : Suppose that $(x_n) \subset \mathcal{T}^{cf}_{\mathbb{H}^2}(S)$ is a sequence that is not contained in any compact subset of $\mathcal{T}^{cf}_{\mathbb{H}^2}(S)$. Suppose, by contradiction, that the image $E_{\mathcal{F}_A}(x_n) \subset (\mathbb{R}_{>0})^{\mathcal{F}_A}$ is bounded, or, in other words, the functions $\{J_\gamma\}_{\gamma\in A}$ are bounded on (x_n), hence the sequence (x_n) converges (up to subsequences) to a point of $\mathrm{Char}(\pi_1(S), SL_{n+1}(\mathbb{R}))$. As $\mathcal{T}^{cf}_{\mathbb{H}^2}(S)$ is closed in $\mathrm{Char}(\pi_1(S), SL_{n+1}(\mathbb{R}))$, we have a contradiction. $\square$

Proposition 101. *Let $\overline{S} = \Sigma^1_1$, whose interior part S is the once punctured torus. Then $\pi_1(S)$ is a free group over two generators, α and β. If $A = \{\alpha, \beta, \alpha\beta\} \subset \pi_1(S)$, then the family $\mathcal{F}_A = \{J_\alpha, J_\beta, J_{\alpha\beta}\}$ is a proper and injective family.*

Proof : First note that the family of functions $\mathcal{F}_A$ generate the ring of coordinates of $\mathrm{Char}(\pi_1(S), SL_{n+1}(\mathbb{R}))$ (see the argument used in the proof of [CS83, prop. 1.4.1]), hence, by the previous lemma, $\mathcal{F}_A$ is a proper family. By [Gu, thm. 1], every function J_γ, with $\gamma \in \pi_1(S)$, can be written as a Laurent polynomial in the indeterminates $J_\alpha, J_\beta, J_{\alpha\beta}$. The conclusion now follows from proposition 88. $\square$

Now we consider the general case. First we need another way for finding proper families.

Proposition 102. *Suppose that $A \subset \pi_1(S)$ has the property that the free homotopy classes of curves in A* **fill up**, *i.e. every free homotopy class of closed curves on the surface has non zero intersection number with at least one of those curves. Then $\mathcal{F}_A = \{J_\gamma\}_{\gamma \in A}$ is proper.*

Proof : We have to show that for every sequence $(x_n) \subset \mathcal{T}^{cf}_{\mathbb{H}^2}(S)$ that is not contained in a compact subset of $\mathcal{T}^{cf}_{\mathbb{H}^2}(S)$, the sequence $(J_\gamma(x_n))_{\gamma \in A}$ is unbounded in $\mathbb{R}^{\mathcal{F}_A}$. Up to subsequences, we can suppose that $(x_n) \longrightarrow x \in \partial_{\mathcal{G}} \mathcal{T}^{cf}_{\mathbb{H}^2}(S)$. Let l be a measured foliation associated to x in Thurston's interpretation. As A is a system that fills up, there exists a $\gamma \in A$ such that $I(l, \gamma) \neq 0$, and, this implies that $J_\gamma(x_n)$ is unbounded (see the proof of proposition 97). Hence $\mathcal{F}_A$ is proper. □

Proposition 103. *There exists an injective family $A \subset \pi_1(A)$, consisting on $9g - 9 + 3b$ elements.*

Proof : There exists $3g - 3 + b$ simple curves on S (denoted by $K_1 \dots K_{3g-3+b}$) that decompose it in $2g - 2 + b$ pair-of-pants, b of them containing a boundary component of S. Let K_i be a curve that is the common boundary of two pair-of-pants whose union will be denoted by P_i. We denote by K_i', K_i'' the two simple closed curves in P_i defined by Thurston in the classification of measured foliation (See [FLP]). Let $A \subset \pi_1(S)$ be the set of the homotopy classes of the curves K_i, K_i' and K_i''. This set fills up, hence $\mathcal{F}_A$ is proper, and the map $\partial \pi_{\mathcal{F}} : \partial_{\mathcal{G}} \mathcal{T}^{cf}_{\mathbb{H}^2}(S) \longrightarrow \partial_{\mathcal{F}_A} \mathcal{T}^{cf}_{\mathbb{H}^2}(S)$ is well defined. The fact that this map is injective from Thurston's classification of measured foliations (see [FLP]). □

5.4.2 The smallest injective families

It may be useful to find a set $A \subset \pi_1(S)$ of minimal cardinality such that $\mathcal{F}_A$ is injective. Such a set can be found with $6g - 5 + 2b$ elements, just one more than the dimension of $\mathcal{T}^{cf}_{\mathbb{H}^2}(S)$. What we need is a set of curves that fill up and such that the map $(I(\cdot, c))_{c \in A}$ from the set of equivalence classes of measured laminations in $\mathbb{R}^{\mathcal{F}_A}$ is injective. If $k = 0$ such a set is described in [Ha2], else it is described in [Ha1]. In the following A will denote a set with these properties.

Let $W = E_{\mathcal{F}_A}(\mathcal{T}^{cf}_{\mathbb{H}^2}(S)) \subset \mathbb{R}^{6g-5+2b}$, a closed semi-algebraic set of dimension less than or equal to the dimension of $\mathcal{T}^{cf}_{\mathbb{H}^2}(S)$, $6g - 6 + 2b$. The boundary of $\partial_{\mathcal{F}_A} \mathcal{T}^{cf}_{\mathbb{H}^2}(S)$ is equal to the boundary ∂W. As $\mathcal{F}_A$ is injective, the boundary ∂W has dimension $6g - 7 + 2b$, and this implies that the dimension of W is exactly $6g-6+2b$. Hence W is an hypersurface in $\mathbb{R}^{6g-5+2b}$, and the cone over its boundary is contained in a tropical hypersurface. In this way we identify the cone over the boundary of the Teichmüller space with a subpolyhedron of a tropical hypersurface in $\mathbb{R}^{6g-5+2b}$.

If $k > 0$ and A is the set described in [Ha1], the map $E_{\mathcal{F}_A}$ is also injective. This may be shown using the fact that the map

$$\mathcal{T}^{cf}_{\mathbb{H}^2}(S) \ni x \longrightarrow (\ell_\gamma(x))_{\gamma\in A} \in \mathbb{R}^A$$

is injective (see [Ha1]), and that $\mathcal{F}_A$ is the composition of this map with cosh.

So we have a semi-algebraic homeomorphism from $\mathcal{T}^{cf}_{\mathbb{H}^2}(S)$ to the closed semi-algebraic hypersurface W.

5.4.3 Description of the piecewise linear structure

We have constructed a piecewise linear structure on the boundary of the Teichmüller spaces $\mathcal{T}^{cf}_{\mathbb{H}^2}(S)$. A piecewise linear structure on $\partial_{\mathcal{G}}\mathcal{T}^{cf}_{\mathbb{H}^2}(S)$ was already known, it was defined by Thurston. See [Pap] for details on this structure.

Theorem 104. *The piecewise linear structure defined above on $\partial_{\mathcal{G}}\mathcal{T}^{cf}_{\mathbb{H}^2}(S)$ is the same as the one discussed in [Pap].*

Proof : In [Pap] the piecewise linear structure is defined using train tracks. An admissible train track on S, is a graph embedded in S satisfying certain conditions. A measure on a train track is a function from the set of the edges in $\mathbb{R}_{>0}$, satisfying, again, certain conditions. There exists an enlargement operation associating to every measured admissible train track a measured foliation f on S, hence a point of $C(\partial_{\mathcal{G}}\mathcal{T}^{cf}_{\mathbb{H}^2}(S))$.

If τ is a fixed train track with n edges, every measure on τ may be identified with a point of $(\mathbb{R}_{>0})^n$, and the subset of all these point is a polyhedral conic subset, that will be denoted by C_τ. The enlargement operation defines a map $\phi_\tau : C_\tau \longrightarrow C(\partial_{\mathcal{G}}\mathcal{T}^{cf}_{\mathbb{H}^2}(S))$. If every connected component of $S\setminus\tau$ is a triangle, the image $\phi_\tau(C_\tau) = V_\tau$ is an open subset of $C(\partial_{\mathcal{G}}\mathcal{T}^{cf}_{\mathbb{H}^2}(S))$. Moreover the map ϕ_τ is an homeomorphism with its image.

The union of the open sets V_τ is the whole $C(\partial_{\mathcal{G}}\mathcal{T}^{cf}_{\mathbb{H}^2}(S))$. So every V_τ is identified with C_τ in such a way that the changes of charts are piecewise linear. This way we have described a piecewise linear atlas for $C(\partial_{\mathcal{G}}\mathcal{T}^{cf}_{\mathbb{H}^2}(S))$, the piecewise linear structure defined by Thurston.

We want to show that the identity map is a piecewise linear map if we endow the domain with the Thurston's piecewise linear structure, and the codomain with the structure defined above. This implies that the two structure are equal.

To show this we need to prove that the maps ϕ_τ are piecewise linear if we endow the codomain with the structure defined above. We choose an injective family $A \subset \pi_1(S)$, and the cone $C(\partial_{\mathcal{F}_A}\mathcal{T}^{cf}_{\mathbb{H}^2}(S))$ is a subset of $\mathbb{R}^{\mathcal{F}_A}$. The coordinates of the map $\phi_\tau : C_\tau \longrightarrow C(\partial_{\mathcal{F}_A}\mathcal{T}^{cf}_{\mathbb{H}^2}(S)) \subset \mathbb{R}^{\mathcal{F}_A}$, can be described as follows, for each element $\gamma \in A$ the corresponding coordinate of ϕ_τ is the function that associate to a measure μ on τ the number $(I(f, \gamma))$, where f is the foliation constructed by the enlargement of the measured train track (τ, μ).

For all $\gamma \in A$ it is easy to see that the corresponding coordinate is piecewise linear. We choose a curve c that is freely homotopic to γ, such that c does not contain any vertex of τ and such that it intersects every edge transversely. Now we define the function $p_\gamma : C_\tau \longrightarrow \mathbb{R}$ as the sum of the measures of all the edges intersected by c, counted with multiplicity. This function is the restriction of a linear function with positive integer coefficients. The coordinate of ϕ_τ corresponding to γ is simply the minimum of all the function p_γ, and locally the minimum may be taken over a finite numbers of linear functions, so the result is a piecewise linear function. $\square$

Chapter 6

Tropical Modules

6.1 Introduction

The points of the Teichmüller space $\mathcal{T}^{cf}_{\mathbb{H}^2}(S)$ and of the space of convex projective structures $\mathcal{T}^{c}_{\mathbb{RP}^n}(M)$, correspond to conjugacy classes of actions of the fundamental group of the manifold on a vector space $\mathbb{R}^{n+1}$, or, equivalently, on a projective space $\mathbb{RP}^n$. In this chapter we introduce the tropical counterparts of these actions, i.e. actions of a group on tropical modules and tropical projective spaces.

There is a naif notion of tropical projective space, the projective quotient of a free module $\mathbb{T}^n$, but these spaces have few invertible projective maps, hence they have few group actions. We give a more general notion of tropical modules and, correspondingly, of tropical projective spaces. We show that these objects have an intrinsic metric, the tropical version of the Hilbert metric, that is invariant for tropical projective maps, and that the topology induced by this metric is contractible. Then we construct a special class of tropical projective spaces by using a generalization of the Bruhat-Tits buildings for SL_n to non-archimedean fields with a surjective real valuation.

In the usual case of a field $\mathbb{F}$ with integral valuation, Bruhat and Tits constructed a polyhedral complex of dimension n with an action of $SL_{n+1}(\mathbb{F})$. In the case $n = 1$, Morgan and Shalen generalized this construction to a field with a general valuation, and they studied these objects using the theory of real trees. We extend this to general n, and we think that the proper structure to study these objects is the structure of tropical projective spaces.

The paper [JSY], developed independently from this work, contains a similar approach to the Bruhat-Tits buildings. Tropical geometry is used there to study the convexity properties of the Bruhat-Tits buildings for $SL_n(\mathbb{F})$, for a field $\mathbb{F}$ with integral valuation.

A brief description of the following sections. In section 6.2 we give some very elementary definitions of semifields, semimodules over a semifield and projective spaces associated with semifields, and we give some examples of semimodules.

In the first subsection of section 6.3 we discuss invertibility of linear maps in $\mathbb{T}^n$ and we introduce the pseudo-inverse function, that will be used in the second subsection, where we consider linear maps f on a vector space $\mathbb{F}^n$ over a non archimedean field $\mathbb{F}$. With every such map f we associate a linear map f^τ on $\mathbb{T}^n$, and we discuss the relations between f^τ and $(f^{-1})^\tau$: globally they are not inverse one of the other, but this happens on a specific "inversion-domain". This will be used in section 6.4, in the description of our generalization of the Bruhat-Tits buildings.

In section 6.4 we define the structure of tropical projective space we put on the generalization of the Bruhat-Tits buildings, and we give a description of this space. Tropical modules $\mathbb{T}^n$ can be seen as the tropicalization of a vector space $\mathbb{F}^n$ over a non-archimedean field $\mathbb{F}$, but this tropicalization depends on the choice of a basis of $\mathbb{F}^n$. Our description with tropical charts, one for each basis of $\mathbb{F}^n$, can be interpreted by thinking the Bruhat-Tits buildings as a tropicalization of $\mathbb{F}^n$ with reference to all possible bases.

In section 6.5 we define in a canonical way a metric on tropical projective spaces making tropical segments geodesics and tropical projective maps 1-Lipschitz. This metric is the transposition to tropical geometry of the Hilbert metric on convex subsets of $\mathbb{RP}^n$. The topology induced by this metric is shown to be contractible. This property is important when we consider an action of a fundamental group of a manifold on a tropical projective space.

6.2 First definitions

6.2.1 Tropical semifields

We need some linear algebra over the tropical semifield. By a **semifield** we mean a quintuple $(S, \oplus, \odot, 0, 1)$, where S is a set, $\oplus$ and $\odot$ are associative and commutative operations $S \times S \longrightarrow S$ satisfying the distributivity law, $0, 1 \in S$ are, respectively, the neutral elements for $\oplus$ and $\odot$. Moreover we require that every element of $S^* = S \setminus \{0\}$ has a multiplicative inverse. We will denote the inverse of a by $a^{\odot -1}$. Given an element $b \neq 0$ we can write $a \oslash b = a \odot b^{\odot -1}$. Note that 0 is never invertible and $\forall s \in S : 0 \odot s = 0$.

A semifield is called **idempotent** if $\forall s \in S : s \oplus s = s$. In this case a partial order relation is defined by

$$a \leq b \Leftrightarrow a \oplus b = b$$

We will restrict our attention to the idempotent semifields such that this partial order is total. In this case $(S \setminus \{0\}, \odot, \leq)$ is an abelian ordered group. Vice versa, given an abelian ordered group $(\Lambda, +, <)$, we add to it an extra element $-\infty$ with the property $\forall \lambda \in \Lambda : -\infty < \lambda$, and we define a semifield:

$$\mathbb{T} = \mathbb{T}_\Lambda = (\Lambda \cup \{-\infty\}, \oplus, \odot, -\infty, 0)$$

with the tropical operations $\oplus, \odot$ defined as

$$a \oplus b = \max(a, b)$$

$$a \odot b = \begin{cases} a + b & \text{if } a, b \in \Lambda \\ -\infty & \text{if } a = -\infty \text{ or } b = -\infty \end{cases}$$

We will use the notation $1_\mathbb{T} = 0$, as the zero of the ordered group is the one of the semifield, and $0_\mathbb{T} = -\infty$. If $a \in \mathbb{T}$ and $a \neq 0_\mathbb{T}$, then $a \odot (-a) = 1_\mathbb{T}$. Hence $-a = a^{\odot -1}$, the **tropical inverse** of a. The order on $\Lambda \cup \{-\infty\}$ induces a topology on $\mathbb{T}$ that makes the operations continuous.

Semifields of the form $\mathbb{T} = \mathbb{T}_\Lambda$ will be called **tropical semifields**. The semifield that in literature is called the tropical semifield is, in our notation, $\mathbb{T}_\mathbb{R}$.

We are interested in tropical semifields because they are the images of valuations. Let $\mathbb{F}$ be a field, Λ an ordered group, and $v : \mathbb{F} \longrightarrow \Lambda \cup \{+\infty\}$ a surjective valuation. Instead of using the valuation, we prefer the **tropicalization map**:

$$\tau : \mathbb{F} \ni z \longrightarrow -v(z) \in \mathbb{T} = \mathbb{T}_\Lambda = \Lambda \cup \{-\infty\}$$

The tropicalization map satisfies the properties of a norm:

1. $\tau(z) = 0_\mathbb{T} \Leftrightarrow z = 0$
2. $\tau(zw) = \tau(z) \odot \tau(w)$
3. $\tau(z + w) \leq \tau(z) \oplus \tau(w)$
4. τ is surjective.

For every element $\lambda \in \mathbb{T}$ we choose an element $t_\lambda \in \mathbb{F}$ such that $\tau(t_\lambda) = \lambda$.

We will denote the valuation ring by $\mathcal{O} = \{z \in \mathbb{F} \mid \tau(z) \leq 1_\mathbb{T}\}$, its unique maximal ideal by $m = \{z \in \mathbb{F} \mid \tau(z) < 1_\mathbb{T}\}$, its residue field by $D = \mathcal{O}/m$ and the projection by $\pi : \mathcal{O} \longrightarrow D$.

Proposition 105. *The map τ 'often' sends $+$ to $\oplus$, i.e.:*

1. $\tau(w_1) \neq \tau(w_2) \Rightarrow \tau(w_1 + w_2) = \tau(w_1) \oplus \tau(w_2)$.
2. *If $\tau(w_1) = \tau(w_2) = \lambda$, then $t_{-\lambda} w_1, t_{-\lambda} w_2 \in \mathcal{O} \setminus m$. In this case* $\pi(t_{-\lambda} w_1) + \pi(t_{-\lambda} w_2) \neq 0 \in D \Rightarrow \tau(w_1 + w_2) = \tau(w_1) \oplus \tau(w_2)$.

Proof : It follows from elementary properties of valuations. □

6.2.2 Tropical semimodules and projective spaces

Definition 106. *Given a semifield S, an S-**semimodule** is a triple $(M, \oplus, \odot, 0)$, where M is a set, $\oplus$ and $\odot$ are operations:*

$$\oplus : M \times M \longrightarrow M \qquad\qquad \odot : S \times M \longrightarrow M$$

$\oplus$ is associative and commutative and $\odot$ satisfies the usual associative and distributive properties of the product by a scalar. We will also require that

$$\forall v \in M : 1 \odot v = v \qquad\qquad \forall v \in M : 0 \odot v = 0$$

Note that the following properties also holds:

$$\forall a \in S : a \odot 0 = 0$$

$$\forall a \in S^* : \forall v \in M : a \odot v = 0 \Rightarrow v = 0$$

The first follows as $a\odot 0\oplus b = a\odot 0\oplus a\odot(a^{-1}\odot b) = a\odot(0\oplus a^{-1}\odot b) = a\odot a^{-1}\odot b = b$. And then the second follows as $0 = a \odot v \Rightarrow 0 = a^{-1} \odot 0 = 1 \odot v = v$.

Most definitions of linear algebra can be given as usual. Let S be a semifield and M a S-semimodule. A **submodule** of M is a subset closed for the operations. If $v_1, \ldots, v_n \in M$, a **linear combination** of them is an element of the form $c_1 \odot v_1 \oplus \cdots \oplus c_n \odot v_n$. If $A \subset M$ is a set, it is possible to define its **spanned submodule** $\mathrm{Span}_S(A)$ as the smallest submodule containing A or, equivalently, as the set of all linear combinations of elements in A. A **linear map** between two S-semimodules is a map preserving the operations. The image of a linear map is a submodule, but (in general) there is not a good notion of kernel.

If S is an idempotent semifield, then M is an idempotent semigroup for $\oplus$. In this case a partial order relation is defined by

$$v \leq w \Leftrightarrow v \oplus w = w$$

Linear maps are monotone with reference to this order.

Let S be a semifield and M be an S-module. The **projective equivalence** relation on M is defined as:

$$x \sim y \Leftrightarrow \exists \lambda \in S^* : x = \lambda \odot y$$

This is an equivalence relation. The **projective space** associated with M may be defined as the quotient by this relation:

$$\mathbb{P}(M) = (M \setminus \{0\})/\sim$$

The quotient map will be denoted by $\pi : M \setminus \{0\} \longrightarrow \mathbb{P}(M)$.

The image by π of a submodule is a **projective subspace**.

If $f : M \longrightarrow N$ is a linear map, we note that $v \sim w \Rightarrow f(v) \sim f(w)$. The linear map induces a map between the associated projective spaces provided that the following condition holds:

$$\{v \in M \mid f(v) = 0\} \subset \{0\}$$

We will denote the induced map as $\overline{f} : \mathbb{P}(M) \longrightarrow \mathbb{P}(N)$. Maps of this kind will be called **projective maps**. The condition does not imply in general that the map is injective. Actually a projective map $\overline{f} : \mathbb{P}(M) \longrightarrow \mathbb{P}(M)$ may be not injective nor surjective in general.

6.2.3 Examples

From now on we will consider only semimodules over a tropical semifield $\mathbb{T} = \mathbb{T}_\Lambda$. The simplest example of $\mathbb{T}$-semimodule is the **free** $\mathbb{T}$-semimodule of **rank** n, i.e. the set $\mathbb{T}^n$ where the semigroup operation is the component wise sum, and the product by a scalar is applied to every component. If $x \in \mathbb{T}^n$ we will write by $x^1, \ldots, x^n$ its components:

$$x = \begin{pmatrix} x^1 \\ \vdots \\ x^n \end{pmatrix}$$

These modules inherit a topology from the order topology of the tropical semifields: the product topology on the free modules and the subspace topology on their submodules. The partial order on these semimodules can be expressed in coordinates as

$$\forall x, y \in \mathbb{T}^n : x \preceq y \Leftrightarrow \forall i : x^i \leq y^i$$

Other examples are the submodules

$$F\mathbb{T}^n = \mathrm{Span}_{\mathbb{T}}((\mathbb{T}^*)^n) = (\mathbb{T}^*)^n \cup \{0_{\mathbb{T}}\} \subset \mathbb{T}^n$$

The projective space associated with $\mathbb{T}^n$ is $\mathbb{P}(\mathbb{T}^n) = \mathbb{TP}^{n-1}$, and the projective space associated with $F\mathbb{T}^n$ is $\mathbb{P}(F\mathbb{T}^n) = F\mathbb{TP}^{n-1}$. We will denote its points with homogeneous coordinates:

$$\pi(x) = [x^1 : x^2 : \cdots : x^n]$$

These projective spaces inherit the quotient topology, and projective maps are continuous for this topology.

$\mathbb{TP}^1 = \mathbb{P}(\mathbb{T}^2)$ can be identified with $\Lambda \cup \{-\infty, +\infty\}$ via the map:

$$\mathbb{TP}^1 \ni [x^1 : x^2] \longrightarrow x^1 - x^2 \in \Lambda \cup \{-\infty, +\infty\}$$

With this identification $\mathbb{TP}^1$ inherits an order: given $a = [a^1 : a^2], b = [b^1 : b^2] \in \mathbb{TP}^1$, we define $a \preceq b \Leftrightarrow a^1 - a^2 \leq b^1 - b^2$. All tropical projective maps

$\mathbb{TP}^1 \longrightarrow \mathbb{TP}^1$ are never increasing or never decreasing with reference to this order. We give a name to three special points: $0_\mathbb{T} = [0_\mathbb{T} : 1_\mathbb{T}] = -\infty, 1_\mathbb{T} = [1_\mathbb{T} : 1_\mathbb{T}] = 0, \infty_\mathbb{T} = [1_\mathbb{T} : 0_\mathbb{T}] = +\infty$.

When $\Lambda = \mathbb{R}$, $\mathbb{T}_\mathbb{R}\mathbb{P}^{n-1}$ may be described as an $(n-1)$-simplex, whose set of vertices is $\{\pi(e_1), \dots, \pi(e_n)\}$ (e_i being the elements of the canonical basis of $\mathbb{T}^n$). Given a set of vertices A, the face with vertices in A is the projective subspace $\pi(\mathrm{Span}_\mathbb{T}(A))$. $F\mathbb{TP}^{n-1}$ is naturally identified with the interior of the simplex $\mathbb{TP}^{n-1}$.

6.3 Linear maps between free semimodules

6.3.1 Tropical matrices

As before let $\mathbb{T} = \mathbb{T}_\Lambda$ be a tropical semifield. Let e_i be the element of $\mathbb{T}^n$ having 1 as the i-th coefficient and 0 as the others. These elements form the **canonical basis** of $\mathbb{T}^n$.

Let $f : \mathbb{T}^n \longrightarrow \mathbb{T}^m$ be a linear map. Then we can define the matrix $A = [f] = (a^i_j)$ as $a^i_j = (f(e_j))^i$. The usual properties of matrices and linear maps hold in this case:

1. $f(e_j)$ is the j-th column of $[f]$: $f(e_j) = \bigoplus_i a^i_j \odot e_i$.
2. If $v \in \mathbb{T}^n$, $(f(v))^i = \bigoplus_j a^i_j \odot v^j$, or $f(v) = \bigoplus_{i,j} a^i_j \odot v^j \odot e_i$.
3. f is surjective $\Leftrightarrow$ the columns of $[f]$ span $\mathbb{T}^m$.
4. There is a binary correspondence between linear maps and matrices with entries in S.
5. The matrix of the composition of two maps is the product matrix, i.e. $[f \circ g] = [f] \odot [g]$, where $(A \odot B)^i_j = \bigoplus_k A^i_k \odot B^k_j$.

The identity matrix, corresponding to the identity map $\mathrm{Id}_\mathbb{T} : \mathbb{T}^n \longrightarrow \mathbb{T}^n$, will be also denoted by $\mathrm{Id}_\mathbb{T} = ((\delta_\mathbb{T})^i_j)$, where

$$(\delta_\mathbb{T})^i_j = \begin{cases} 1_\mathbb{T} & \text{if } i = j \\ 0_\mathbb{T} & \text{if } i \neq j \end{cases}$$

A linear map $f : \mathbb{T}^n \longrightarrow \mathbb{T}^m$ induces a linear map $f : F\mathbb{T}^n \longrightarrow F\mathbb{T}^m$ by restriction, provided that no element in $F\mathbb{T}^n$ is mapped outside $F\mathbb{T}^m$, i.e. if every row of the matrix $[f]$ contains a non-zero element.

Projective maps $\overline{f} : \mathbb{TP}^{n-1} \longrightarrow \mathbb{TP}^{m-1}$ are induced by matrices mapping no non-zero vector to zero. These are precisely the matrices such that every column contains a non-zero element.

Tropical linear maps are very seldom surjective. This depends on the following property:

$$\mathrm{Span}_\mathbb{T}(v_1, \dots, v_m) = \mathbb{T}^m \Leftrightarrow \forall i = 1, \dots, m : \exists a \in \mathbb{T}^* : a \odot e_i \in \{v_1, \dots, v_m\}$$

Hence a tropical linear map is surjective if and only if it has, among its columns, all the elements of the canonical basis of the codomain.

Let $f : \mathbb{T}^n \longrightarrow \mathbb{T}^m$ be a linear map, with matrix $[f] = (a^i_j)$. Suppose that every column of $[f]$ contains a non-zero element. We will denote by $f^{pi} : \mathbb{T}^m \longrightarrow \mathbb{T}^n$ the map defined by:

$$(f^{pi}(y))^j = \min_i (y^i - a^i_j)$$

(in the previous formula, by $-0_{\mathbb{T}}$ we mean an element greater than every other element in $\mathbb{T}$. This value is never the minimum, thanks to the condition on the columns). In [CGQ04] this map is called residuated map.

Theorem 107. *Let $y \in \mathbb{T}^m$. Then $y \in \operatorname{Im} f$ if and only if exists a sequence $\epsilon : \{1, \dots, m\} \longrightarrow \{1, \dots, n\}$ such that*

$$\forall k = 1, \dots, m : y^k - a^k_{\epsilon_k} = (f^{pi}(y))^{\epsilon_k}$$

Moreover we have

$$f^{-1}(y) = \bigcup_{\epsilon \text{ as before}} \left\{ x \in \mathbb{T}^n \mid \begin{array}{l} x \preceq f^{pi}(y) \\ \forall k = 1, \dots, m : x^{\epsilon_k} = (f^{pi}(y))^{\epsilon_k} \end{array} \right\}$$

This implies that $f^{-1}(y)$ is a single point if and only if every function ϵ as before is surjective.

The function f^{pi} plays the role of a pseudo-inverse function, as it sends every point of the image in one of its pre-images, in a continuous way. It has the following properties:

1. $\forall x \in \mathbb{T}^n : \forall y \in \mathbb{T}^m : \big(x \preceq f^{pi}(y) \Leftrightarrow f(x) \preceq y\big)$.
2. $\forall x \in \mathbb{T}^n : x \preceq f^{pi}(f(x))$
3. $\forall y \in \mathbb{T}^m : f(f^{pi}(y)) \preceq y$
4. $\forall y \in \operatorname{Im} f : f(f^{pi}(y)) = y$
5. $\forall x \in \operatorname{Im} f^{pi} : f^{pi}(f(x)) = x$
6. $f_{|\operatorname{Im} f^{pi}} : \operatorname{Im} f^{pi} \longrightarrow \operatorname{Im} f$ *is bijective, with inverse* f^{pi}.

Proof: The point y is in the image if and only if exists $x \in \mathbb{T}^n$ such that $f(x) = y$. Then

$$f(x) = y \;\Leftrightarrow\; \forall i : \bigoplus_j (a^i_j \odot x^j) = y^i \;\Leftrightarrow\; \left\{ \begin{array}{l} \forall i, j : a^i_j + x^j \le y^i \\ \forall i : \exists j : a^i_j + x^j = y^i \end{array} \right. \;\Leftrightarrow$$

$$\left\{ \begin{array}{l} \forall i, j : x^j \le y^i - a^i_j \\ \forall i : \exists j : x^j = y^i - a^i_j \end{array} \right. \;\Leftrightarrow\; \left\{ \begin{array}{l} \forall j : x^j \le \min_i (y^i - a^i_j) \\ \forall i : \exists j : x^j = y^i - a^i_j \end{array} \right.$$

Then

$$y \in \operatorname{Im} f \ \Leftrightarrow \exists \epsilon : \forall k : y^k - a^k_{\epsilon_k} = \min_i (y^i - a^i_{\epsilon_k})$$

In this case $x^{\epsilon_k} = y^k - a^k_{\epsilon_k}$. All the claims of the theorem follows from the calculations above. □

6.3.2 Simple tropicalization of linear maps

Let $\mathbb{F}$ be a valued field, with tropicalization map $\tau : \mathbb{F} \longrightarrow \mathbb{T}$. An $\mathbb{F}$-vector space $\mathbb{F}^n$ may be tropicalized through the componentwise tropicalization map, again denoted by $\tau : \mathbb{F}^n \longrightarrow \mathbb{T}^n$.

Let $f : \mathbb{F}^n \longrightarrow \mathbb{F}^m$ be a linear map, expressed by a matrix $[f] = (a^i_j)$. Its tropicalization is the map $f^\tau : \mathbb{T}^n \longrightarrow \mathbb{T}^m$ defined by the matrix $[f^\tau] = (\alpha^i_j) = (\tau(a^i_j))$.

Proposition 108. *The following properties hold:*

1. $\forall z \in \mathbb{F}^n : \tau(f(z)) \preceq f^\tau(\tau(z))$.
2. $\forall x \in \mathbb{T}^n : \exists z \in \mathbb{F}^n : \tau(z) = x$ *and* $\tau(f(z)) = f^\tau(x)$.

□

Let $A \in GL_n(\mathbb{F})$ be an invertible matrix. Its tropicalization $\alpha = A^\tau : \mathbb{T}^n \longrightarrow \mathbb{T}^n$ (i.e. $\alpha = (\alpha^i_j) = (\tau(a^i_j)))$ is, in general, not invertible. Anyway it has the property that every column and every row contains a non-zero element, hence it has a pseudo-inverse function, and it induces a linear map $F\mathbb{T}^n \longrightarrow F\mathbb{T}^n$, and projective maps $\mathbb{TP}^{n-1} \longrightarrow \mathbb{TP}^{n-1}$ and $F\mathbb{TP}^{n-1} \longrightarrow F\mathbb{TP}^{n-1}$.

Now let $B = A^{-1}$, the inverse of A. We will write $\beta = B^\tau$. We would like to see β as an inverse of α, but this is impossible, as α is not always invertible.

Proposition 109. *The following statements hold*

1. $\forall i, j : (\alpha \odot \beta)^i_j \geq (\delta_\mathbb{T})^i_j$ *and* $(\beta \odot \alpha)^i_j \geq (\delta_\mathbb{T})^i_j$.
2. $\forall x \in \mathbb{T}^n : x \preceq \alpha(\beta(x))$ *and* $y \preceq \beta(\alpha(y))$.
3. $\forall x \in \mathbb{T}^n : \alpha^{pi}(x) \preceq \beta(x)$.
4. $\forall x \in \mathbb{T}^n : \alpha(\beta(x)) = x \Leftrightarrow \beta(x) = \alpha^{pi}(x)$

Proof :

1. It follows from: $AB = \mathrm{Id}$, $BA = \mathrm{Id}$, $\tau(z_1 + z_2) \leq \tau(z_1) \oplus \tau(z_2)$.
2. It follows from the previous statement.

3. This is equivalent to $\forall i : \max_j(\beta^i_j + x^j) \geq \min_j(x_j - \alpha^j_i)$, i.e. $\forall i : \exists k, h : \beta^i_k + x^k \geq x^h - \alpha^h_i$. This always holds as, from the first statement, we know that $\max_k(\beta^i_k + \alpha^k_i) = (\beta \odot \alpha)^i_i \geq 1_{\mathbb{T}}$, hence $\forall i : \exists k : \beta^i_k + x^k \geq x^k - \alpha^k_i$.

4. From theorem 107, part 1, we know that $\alpha(\beta(x)) \preceq x \Leftrightarrow \beta(x) \preceq \alpha^{pi}(x)$. The reversed inequalities always holds.

□

If α and β are tropicalizations of two maps $A, B \in GL_n(\mathbb{F})$ such that $A^{-1} = B$, we will call **inversion domain** the set $D_{\alpha\beta} = \{x \in \mathbb{T}^n \mid \alpha(\beta(x)) = x\}$.

Proposition 110. *The inversion domains have this name because of the following property: $D_{\beta\alpha} = \beta(D_{\alpha\beta})$, $D_{\alpha\beta} = \alpha(D_{\beta}\alpha)$ and $\beta_{|D_{\alpha\beta}} : D_{\alpha\beta} \longrightarrow D_{\beta\alpha}$ is bijective with inverse $\alpha_{|D_{\beta\alpha}} : D_{\beta\alpha} \longrightarrow D_{\alpha\beta}$.*

The set $D_{\alpha\beta}$ is a tropical submodule, and we can write explicit equations for it:

$$D_{\alpha\beta} = \{x \in \mathbb{T}^n \mid \forall h, k : x^h - x^k \geq (\alpha \odot \beta)^h_k\}$$

As a consequence if $A \in GL_n(\mathcal{O})$, then $D_{\alpha\beta} \neq \emptyset$. Note that the matrices α and β are not one the inverse of the other, but, in the hypothesis $D_{\alpha\beta} \neq \emptyset$, then $\forall i : (\alpha \odot \beta)^i_i = 1_{\mathbb{T}}$.

The map $\beta_{|D_{\alpha\beta}}$ is the composition of a permutation of coordinates and a tropical dilatation: there exists a diagonal matrix d and a permutation of coordinates σ such that $(\sigma \circ d \circ \beta)_{|D_{\alpha\beta}} = Id_{|D_{\alpha\beta}} : D_{\alpha\beta} \longrightarrow D_{\alpha\beta}$. □

6.4 Tropical projective structure on Bruhat-Tits buildings

6.4.1 Definition

Given a non-archimedean field $\mathbb{F}$ with a surjective real valuation, we are going to construct a family of tropical projective spaces we will call $P^{n-1}(\mathbb{F})$, or simply P^{n-1} when the field is well understood. This family arises as a generalization of the Bruhat-Tits buildings for SL_n to non-archimedean fields with surjective real valuation. In the usual case of a field with integral valuation, Bruhat and Tits constructed a polyhedral complex of dimension $n - 1$ with an action of $SL_n(\mathbb{F})$. In the case $n = 2$, Morgan and Shalen generalized this construction to a field with a general valuation, and they studied these objects using the theory of real trees. We want to extend this to general n, and we think that the proper structure to study these objects is the structure of tropical projective spaces.

Let $V = \mathbb{F}^n$, an $\mathbb{F}$-vector space of dimension n and an infinitely generated $\mathcal{O}$-module. We consider the natural action $GL_n(\mathbb{F}) \times V \longrightarrow V$.

Definition 111. *An $\mathcal{O}$-**lattice** of V is an $\mathcal{O}$-finitely generated $\mathcal{O}$-submodule of V.*

Proposition 112. *Let L be an $\mathcal{O}$-finitely generated $\mathcal{O}$-submodule of V. Then every minimal set of generators is $\mathbb{F}$-linearly independent, hence L is free.*

Proof : Let $\{e_1, \dots, e_m\}$ be a minimal set of generators of L. Suppose they are not $\mathbb{F}$-independent. Then there exist $a_1, \dots, a_m \in \mathbb{F}$ s.t. $\sum a_i e_i = 0$. We may suppose $\tau(a_1) \leq \dots \leq \tau(a_m)$. There exist elements $b_1, \dots, b_m \in \mathcal{O}$ s.t. $a_i = b_i a_m$. Hence $a_m(\sum b_i e_i) = 0 \Rightarrow e_m = \sum_1^n b_i e_i$ with $b_1, \dots, b_{m-1} \in \mathcal{O}$. They can't be minimal.

An element of L is an $\mathcal{O}$-linear combination of $\{e_1, \dots, e_m\}$ because they are generators, and the linear combination is unique because they are $\mathbb{F}$-independent. Hence L is free. $\square$

If L is a finitely generated $\mathcal{O}$-submodule of V, its rank is a number from 0 to n.

Definition 113. *A **maximal** $\mathcal{O}$-**lattice** is an $\mathcal{O}$-lattice of rank n.*

We denote by $U^n(\mathbb{F})$ (or simply U^n) the set of all $\mathcal{O}$-lattices of $V = \mathbb{F}^n$, and by $FU^n(\mathbb{F})$ (or simply FU^n) the subset of all maximal $\mathcal{O}$-lattices and the $\mathcal{O}$-lattice $\{0\}$.

U^n and FU^n can be turned in $\mathbb{T}$-semimodules by means of the following operations:

$\oplus : U^n \times U^n \longrightarrow U^n \qquad L \oplus M = \mathrm{Span}_{\mathcal{O}}(L \cup M)$

$\odot : \mathbb{T} \times U^n \longrightarrow U^n \qquad x \odot L = zL$, where $z \in \mathbb{F}, \tau(z) = x$

The associated tropical projective spaces will be denoted by $\mathbb{P}(U^n(\mathbb{F})) = P^{n-1}(\mathbb{F})$ and $\mathbb{P}(FU^n(\mathbb{F})) = FP^{n-1}(\mathbb{F})$. We will simply write P^{n-1} and FP^{n-1} when the field $\mathbb{F}$ is understood.

As we said there is a natural action $GL_n(\mathbb{F}) \times V \longrightarrow V$. Every element $A \in GL_n(\mathbb{F})$ sends $\mathcal{O}$-lattices in $\mathcal{O}$-lattices, hence we have an induced action $GL_n(\mathbb{F}) \times U^n \longrightarrow U^n$. This action preserves the rank of a lattice, and in particular it sends FU^n in itself. Among the $\mathcal{O}$-lattices with the same rank this action is transitive, for example there exist an $A \in GL_n(\mathbb{F})$ sending every maximal $\mathcal{O}$-lattice of V in the standard lattice $\mathcal{O}^n \subset V$.

Hence the group $SL_n(\mathbb{F})$ acts naturally on U^n and FU^n by tropical linear maps and on P^{n-1} and FP^{n-1} by tropical projective maps.

6.4.2 Description

Let $\mathcal{E} = (e_1, \dots, e_n)$ be a basis of V. We denote by $\varphi_{\mathcal{E}} : \mathbb{T}^n \longrightarrow U^n$ the map:

$$\varphi_{\mathcal{E}}(y) = \varphi_{\mathcal{E}}(y^1, \dots, y^n) = I_{y^1} e_1 + \dots + I_{y^n} e_n = \mathrm{Span}_{\mathcal{O}}(t_{y^1} e_1, \dots t_{y^n} e_n)$$

Proposition 114. *Let $< e_1, \ldots, e_m >$ be a $\mathcal{O}$-basis of an $\mathcal{O}$-lattice L, and let $p_i \in \mathbb{F}$. Then:*

1. $p_i e_i \in L \Leftrightarrow p_i \in \mathcal{O} \Leftrightarrow \tau(p_i) \leq 0$.
2. $< p_1 e_1, \ldots p_n e_n >$ *is an $\mathcal{O}$-basis of $L \Leftrightarrow p_i \in \mathcal{O} \setminus m \Leftrightarrow \tau(p_i) = 0$.*

Proof: It follows from the properties of valuations. □

This proposition implies that $\varphi_{\mathcal{E}}$ is injective and $\varphi_{\mathcal{E}}(F\mathbb{T}^n) \subset FU^n$. For every basis $\mathcal{E}$ we have a different map $\varphi_{\mathcal{E}}$. The union of the images of all these maps is the whole U^n, and the union of all the sets $\varphi(F\mathbb{T}^n)$ is equal to FU^n. We will call the maps $\varphi_{\mathcal{E}}$ **tropical charts** for U^n. Theorem 117 will justify this name.

Note that the charts respect the partial order relations on $\mathbb{T}^n$ and on U^n:

$$x \preceq y \Leftrightarrow \varphi(x) \subset \varphi(y)$$

Lemma 115. *Let $L, M \subset V$ be two $\mathcal{O}$-lattices, and suppose that L is maximal. Then there is a basis $v_1, \ldots, v_n$ of L and scalars $a_1, \ldots, a_n \in \mathbb{F}$ such that $a_1 v_1, \ldots a_n v_n$ is a basis of M.*

Proof: Fix a basis $e_1, \ldots, e_n$ of V such that $L = \mathrm{Span}_{\mathcal{O}}(e_1, \ldots, e_n)$. Let $f_1, \ldots, f_n$ be a basis of M. For every vector f_i there is a scalar $b_i \in \mathbb{F}$ such that $b_i f_i \in L$. Then, if b_i is the one with maximal valuation, $b_i M \subset L$. The thesis follows by applying [MS84, corol. II.3.2] to the $\mathcal{O}$-modules L and $b_i M$. □

Corollary 116. *Given two points $x, y \in U^n$, there is a tropical chart containing both of them in its image.* □

Given two bases $\mathcal{E} = (e_1, \ldots, e_n)$ and $\mathcal{F} = (f_1, \ldots, f_n)$, we have two charts $\varphi_{\mathcal{E}}, \varphi_{\mathcal{F}}$. We want to study the intersection of the images.

We put $I = \varphi_{\mathcal{E}}(\mathbb{T}^n) \cap \varphi_{\mathcal{F}}(\mathbb{T}^n)$, $I_{\mathcal{E}} = \varphi_{\mathcal{E}}^{-1}(I)$, $I_{\mathcal{F}} = \varphi_{\mathcal{F}}^{-1}(I)$. We want to describe the sets $I_{\mathcal{F}}, I_{\mathcal{E}}$ and the **transition function**: $\varphi_{\mathcal{F}\mathcal{E}} = \varphi_{\mathcal{F}}^{-1} \circ \varphi_{\mathcal{E}} : I_{\mathcal{E}} \longrightarrow I_{\mathcal{F}}$.

The transition matrices between $\mathcal{E}$ and $\mathcal{F}$ are denoted by $A = (a^i_j), B = (b^i_j) \in GL_n(\mathbb{F})$:

$$\forall j : e_j = \sum_i a^i_j f_i \qquad \forall j : f_j = \sum_i b^i_j e_i \qquad A = B^{-1}$$

We will write $\alpha = A^\tau$ and $\beta = B^\tau$, i.e. $\alpha = (\alpha^i_j) = (\tau(a^i_j))$, $\beta = (\beta^i_j) = (\tau(b^i_j))$.

Theorem 117. [Description of the tropical charts] *We have that $I_{\mathcal{F}} = D_{\alpha\beta}$ and $I_{\mathcal{E}} = D_{\beta\alpha}$, the inversion domains described in proposition 110. Moreover $\varphi_{\mathcal{F}\mathcal{E}} = \alpha_{|I_{\mathcal{E}}}$ and $\varphi_{\mathcal{E}\mathcal{F}} = \beta_{|I_{\mathcal{F}}}$, the tropicalizations of the transition matrices.*

Proof: First, we need to prove the following two assertions:

1. $\varphi_{\mathcal{E}}(y) \subset \varphi_{\mathcal{F}}(x) \Leftrightarrow \alpha(y) \preceq x$ and $\varphi_{\mathcal{F}}(x) \subset \varphi_{\mathcal{E}}(y) \Leftrightarrow \beta(x) \preceq y$.

2. $\varphi_{\mathcal{F}}(x) = \varphi_{\mathcal{E}}(y) \Leftrightarrow x = \alpha(y)$ and $y = \beta(x)$.

Let $w = \sum_i w^i f_i \in \mathbb{F}^n$. Then:
$w \in \mathrm{Span}_{\mathcal{O}}(f_1, \dots, f_n) \Leftrightarrow \forall i : w^i \in \mathcal{O} \Leftrightarrow \forall i : \tau(w^i) \leq 1_{\mathbb{T}}$
$t_y w \in \varphi_{\mathcal{F}}(x) \Leftrightarrow \forall i : \frac{t_y w^i}{t_{x^i}} \in \mathcal{O} \Leftrightarrow \forall i : \tau(w^i) \leq x^i - y$
$\varphi_{\mathcal{E}}(y) \subset \varphi_{\mathcal{F}}(x) \Leftrightarrow \forall j : t_{y^j} e_j \in \varphi_{\mathcal{F}}(x) \Leftrightarrow \forall j, i : \tau(a^i_j) \leq x^i - y^j$
$\varphi_{\mathcal{F}}(x) \subset \varphi_{\mathcal{E}}(y) \Leftrightarrow \forall j, i : \tau(b^i_j) \leq y^i - x^j$
Then we have:

$$\varphi_{\mathcal{E}}(y) = \varphi_{\mathcal{F}}(x) \Leftrightarrow \forall j, i : \tau(b^i_j) \leq y^i - x^j \leq -\tau(a^j_i)$$

$$\Leftrightarrow \forall j, i : \tau(b^i_j) + x^j \leq y^i \leq -\tau(a^j_i) + x^j$$

$$\Leftrightarrow \forall i : \max_j(\tau(b^i_j) + x^j) \leq y^i \leq \min_j(-\tau(a^j_i) + x^j)$$

$$\Leftrightarrow \forall i : \bigoplus_j (\beta^i_j \odot x^j) \leq y^i \leq \min_j (x^j - \alpha^j_i) \Leftrightarrow \forall i : (\beta(x))^i \leq y^i \leq (\alpha^{pi}(x))^i$$

The map $\varphi_{\mathcal{E}}$ is injective, hence, given a fixed x, if an y satisfying the last condition exists, it has to be unique. Then the interval in which its coordinates are free to vary must degenerate to a single point. Then we have:

$$\varphi_{\mathcal{E}}(y) = \varphi_{\mathcal{F}}(x) \Leftrightarrow \beta(x) = y = \alpha^{pi}(x)$$

We can prove the symmetric equalities reversing the roles of $\mathcal{E}$ and $\mathcal{F}$.
Now we look at $\varphi^{-1}_{\mathcal{F}}(I) = \{x \in \mathbb{T}^n \mid \exists y \in \mathbb{T}^n : \varphi_{\mathcal{E}}(y) = \varphi_{\mathcal{F}}(x)\}$. We have

$$x \in \varphi^{-1}_{\mathcal{F}}(I) \Leftrightarrow \exists y : \forall i : \max_j(\beta^i_j + x^j) \leq y^i \leq \min_j(-\alpha^j_i + x^j)$$

$$\Leftrightarrow \forall i : \max_j(\beta^i_j + x^j) \leq \min_j(-\alpha^j_i + x^j) \Leftrightarrow \forall i, k, h : \beta^i_k + x^k \leq -\alpha^h_i + x^h$$

$$\Leftrightarrow \forall i, k, h : x^h - x^k \geq \beta^i_k + \alpha^h_i \Leftrightarrow \forall k, h : x^h - x^k \geq \oplus_i(\alpha^h_i \odot \beta^i_k)$$

□

6.5 Tropical projective spaces as metric spaces

6.5.1 Finitely generated semimodules

Free semimodules have the usual universal property: let M be a $\mathbb{T}$-semimodule, and $v_1, \dots, v_n \in M$. Then there is a linear map:

$$\begin{array}{ccc} \mathbb{T}^n & \longrightarrow & \mathrm{Span}_{\mathbb{T}}(v_1, \ldots, v_n) \\ c & \longrightarrow & c^1 \odot v_1 \oplus \cdots \oplus c^n \odot v_n \end{array}$$

This map sends e_i in v_i and its image is $\mathrm{Span}_{\mathbb{T}}(v_1, \ldots, v_n)$.

Hence every finitely generated $\mathbb{T}$-semimodule is the image of a free $\mathbb{T}$-semimodule.

In the following we will need some properties of finitely generated semimodules over $\mathbb{T}_{\mathbb{R}}$, so for this section we will suppose $\mathbb{T} = \mathbb{T}_{\mathbb{R}}$.

First we want to discuss a pathological example we prefer to neglect. Consider the following equivalence relation on $\mathbb{T}^2$:

$$(x^1, x^2) \sim (y^1, y^2) \Leftrightarrow \begin{cases} x^1 < x^2, y^1 < y^2 \text{ and } x^2 = y^2 \\ or \\ x^1 \geq x^2, y^1 \geq y^2 \text{ and } x^1 = y^1 \end{cases}$$

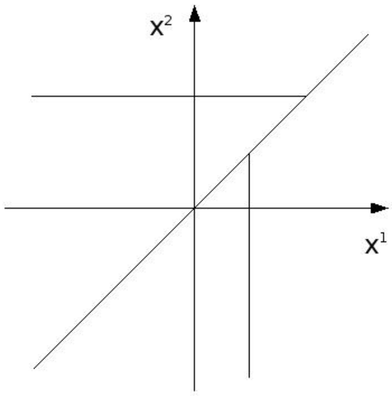

Figure 6.1: Two examples of equivalence classes for the relation defining the quotient module B: $\{x^2 = 2, x^1 < 2\}$ and $\{x^1 = 1, x^2 \leq x^1\}$.

The quotient for this relation will be denoted by B. If $a \sim a'$ and $b \sim b'$, then $a \oplus b = a' \oplus b'$ and $\lambda \odot a = \lambda \odot a'$. Hence the operations $\oplus$, $\odot$ induces operations on B, turning it in a finitely generated $\mathbb{T}$-semimodule. We will denote the equivalence classes in the following way: if (x^1, x^2) satisfies $x^1 < x^2$ we will denote its class as $[(\cdot, x^2)]$, if $x^1 \geq x^2$ we will denote its class as $[(x^1, \cdot)]$. The $\odot$ operation act as

$$\lambda \odot [(\cdot, x^2)] = [(\cdot, \lambda \odot x^2)]$$

and analogously for the other classes. The $\oplus$ operation acts as

$$[(\cdot, x^2)] \oplus [(\cdot, y^2)] = [(\cdot, x^2 \oplus y^2)]$$

$$[(x^1,\cdot)]\oplus[(y^1,\cdot)]=[(x^1\oplus y^1),\cdot]$$
$$[(x^1,\cdot)]\oplus[(\cdot,x^2)]=[(x^1,x^2)]$$

If we put on the quotient a topology making the projection continuous, then the point $[(x^1,\cdot)]$ is not closed, as its closure must contain the point $[(\cdot,x^1)]$.

We define a $\mathbb{T}$-semimodule to be **separated** if it does not contain any submodule isomorphic to B. We will see in the following section that every separated $\mathbb{T}$-semimodule has a natural metrizable topology making all linear maps continuous. Examples of separated $\mathbb{T}$-semimodules are all free semimodules (as there exists no submodule in $\mathbb{T}^n$ whose associated projective space has exactly two points) and the semimodules U^n (as every two points in U^n are in the image of the same tropical chart, hence in a submodule isomorphic to $\mathbb{T}^n$).

Lemma 118. *Let M be a $\mathbb{T}$-semimodule and let $f:\mathbb{T}^2\longrightarrow M$ be a linear map such that $f\begin{pmatrix} x^1\\ x^2\end{pmatrix}=f\begin{pmatrix} y^1\\ x^2\end{pmatrix}=m$ when $y^1\leq x^1$. Then $\forall y\leq x^1$: $f\begin{pmatrix} y\\ x^2\end{pmatrix}=m$.*

Proof : Case 1): If $y^1\leq y\leq x^1$, then $\begin{pmatrix} y^1\\ x^2\end{pmatrix}\preceq\begin{pmatrix} y\\ x^2\end{pmatrix}\preceq\begin{pmatrix} x^1\\ x^2\end{pmatrix}$. Linear maps are monotone with reference to $\preceq$, hence $m\preceq f\begin{pmatrix} y\\ x^2\end{pmatrix}\preceq m$.

Case 2): If $y=y^1-(x^1-y^1)$, then consider the points $a=(y^1-x^1)\odot\begin{pmatrix} x^1\\ x^2\end{pmatrix}$ and $b=(y^1-x^1)\odot\begin{pmatrix} y^1\\ x^2\end{pmatrix}$. We have $f(a)=f(b)=(y^1-x^1)\odot m$. Then

$$f\begin{pmatrix} y\\ x^2\end{pmatrix}=f\left(\begin{pmatrix} y\\ x^2\end{pmatrix}\oplus b\right)=f\begin{pmatrix} y\\ x^2\end{pmatrix}\oplus f(b)=$$
$$=f\begin{pmatrix} y\\ x^2\end{pmatrix}\oplus f(a)=f\left(\begin{pmatrix} y\\ x^2\end{pmatrix}\oplus a\right)=f\begin{pmatrix} y^1\\ x^2\end{pmatrix}$$

Case 3): General case. Iterating the proof of case 2 we can prove the lemma for $y=y^1-n(x^1-y^1)$. Then by case 1 we can extend the result to every y. □

Proposition 119. *Let M be a $\mathbb{T}$-semimodule and let $f:\mathbb{TP}^1\longrightarrow\mathbb{P}(M)$ be a tropical projective map. If f is not injective there are two points $x\prec y\in\mathbb{TP}^1$ such that $f(x)=f(y)=p\in\mathbb{P}(M)$. Then either $\forall z\prec y: f(z)=p$ or $\forall z\succ x: f(z)=p$.*

Proof : The map f is associated with a map $\bar{f}:\mathbb{T}^2\longrightarrow M$. There exists lifts $\bar{x},\bar{y}\in\mathbb{T}^2$ such that $\bar{f}(\bar{x})=\bar{f}(\bar{y})=\bar{p}$. Now:

Case 1) If $\bar{x}\preceq\bar{y}$ then one of their coordinates is equal. Else there is a scalar $\lambda<1_{\mathbb{T}}$ such that $x\preceq\lambda\odot y$, and $\bar{p}\preceq\lambda\odot\bar{p}$, a contradiction. Then we can apply the previous lemma, and we have that $\forall z\prec y: f(z)=p$.

Case 2) If $\bar{y}\preceq\bar{x}$ as before we have $\forall z\succ x: f(z)=p$.

Case 3) If they are not comparable, then both are minor than their sum, $\bar{x}\oplus\bar{y}$, and $\bar{f}(\bar{x}\oplus\bar{y})=p$. Then, by previous cases we have that $\forall z\in\mathbb{TP}^1: f(z)=p$. □

Corollary 120. *Let M be a $\mathbb{T}$-semimodule and let $f : \mathbb{TP}^1 \longrightarrow \mathbb{P}(M)$ be a tropical projective map. The sets $f^{-1}(f(0_\mathbb{T}))$ and $f^{-1}(f(\infty_\mathbb{T}))$ are, respectively, an initial and a final segment for the order of $\mathbb{TP}^1$. If M is separated, then these segments are closed segments. On the complement of their union the map is injective.* $\square$

Let M be a separated $\mathbb{T}$-semimodule, $\bar{f} : \mathbb{T}^n \longrightarrow M$ be a linear map and $f : \mathbb{TP}^{n-1} \longrightarrow \mathbb{P}(M)$ be the induced projective map. As usual we denote by $e_1, \ldots, e_n$ the points of the canonical basis of $\mathbb{T}^n$, and we pose $v_i = \bar{f}(e_i) \in M$. We want to describe the set $V_i = f^{-1}(\pi(v_i))$. It is enough to describe V_1. As $\mathrm{Span}_\mathbb{T}(e_j, e_1)$ is isomorphic to $\mathbb{T}^2$, we know that $S_j = V_1 \cap \pi(\mathrm{Span}_\mathbb{T}(e_j, e_1))$ is a closed initial segment of $\pi(\mathrm{Span}_\mathbb{T}(e_j, e_1))$, with extreme point $\pi(w_j)$. We can suppose that $w_j = a_j e_j + e_1$.

Lemma 121. *The set V_1 is*

$$\pi(\{e_1 \oplus b_2 \odot e_2 \oplus \cdots \odot b_n \oplus e_n \mid b_i \leq a_i\})$$

Hence there is a point $h_1 = e_1 \oplus a_2 \odot e_2 \oplus \cdots \odot a_n \oplus e_n$ such that $\pi(h_1)$ is an extremal point of V_1.

The restriction of $\bar{f}$ to the submodule $\mathrm{Span}_\mathbb{T}(h_i, h_j)$ *is injective.* $\square$

6.5.2 Definition of the metric

As we saw in subsection 2.3.3, every properly convex subset of $\mathbb{RP}^n$ has a canonical metric invariant by projective automorphisms, the Hilbert metric.

We can give an analogous definition for separated tropical projective spaces over $\mathbb{T}_\mathbb{R}$. In the following we will assume $\Lambda = \mathbb{R}$ and $\mathbb{T} = \mathbb{T}_\mathbb{R}$. If M is a separated $\mathbb{T}$-module there is a canonical way for defining a distance $d : \mathbb{P}(M) \times \mathbb{P}(M) \longrightarrow \mathbb{R} \cup \{+\infty\}$. This distance differs from ordinary distances as it can take the value $+\infty$, but has the other properties of a distance (non degeneracy, symmetry, triangular inequality). If $f : \mathbb{P}(M) \longrightarrow \mathbb{P}(N)$ is a projective map, then $d(f(x), f(y)) \leq d(x, y)$, and if $S \subset M$ is such that $f_{|S}$ is injective, then $f_{|S}$ is an isometry.

This metric can be defined searching for a tropical analogue of the cross ratio. In $\mathbb{RP}^1$ the cross ratio can be defined by the identity $[0, 1, z, \infty] = z$ and the condition of being a projective invariant. Or equivalently if A is the (unique) projective map satisfying $A(0) = a, A(1) = b, A(\infty) = d$, then $[a, b, c, d] = A^{-1}(c)$. In this form the definition can be transposed to the tropical case.

Let $\mathbb{T}$ be a tropical semifield and let $a = [a^1 : a^2], b = [b^1 : b^2], c = [c^1 : c^2], d = [d^1 : d^2] \in \mathbb{TP}^1 = \mathbb{P}(\mathbb{T}^2)$ be points such that $a \prec b \prec c \prec d$. There is a unique tropical projective map A satisfying $A(0_\mathbb{T}) = a, A(1_\mathbb{T}) = b, A(\infty_\mathbb{T}) = d$. This map is described by the matrix

$$\begin{pmatrix} a^2 \odot b^1 \odot d^1 & a^1 \odot b^2 \odot d^1 \\ a^2 \odot b^1 \odot d^2 & a^2 \odot b^2 \odot d^1 \end{pmatrix}$$

Given an $x \in \mathbb{T}, x \geq 1_{\mathbb{T}}$, we have that

$$A([x : 1_{\mathbb{T}}]) = \begin{cases} [b^1 + x : b^2] & \text{if } x < (d^1 - d^2) - (b^1 - b^2) \\ d & \text{else} \end{cases}$$

The point $A^{-1}(c)$ is then $[(c^1 - c^2) - (b^1 - b^2) : 1]$. Then we can define this point of $\mathbb{TP}^1$ as the cross-ratio of $[a, b, c, d]$. This value depends only on the central points b, c, and it is invariant by every tropical projective map $B : \mathbb{TP}^1 \longrightarrow \mathbb{TP}^1$ that is injective on the interval $[b, c]$.

Consider a tropical projective map $B : \mathbb{TP}^1 \longrightarrow \mathbb{TP}^1$ such that $B(0_{\mathbb{T}}) = b$ and $B(\infty_{\mathbb{T}}) = c$. This map is described by a matrix of the form:

$$\begin{pmatrix} \mu \odot c^1 & \lambda \odot b^1 \\ \mu \odot c^2 & \lambda \odot b^2 \end{pmatrix}$$

The inverse images $B^{-1}(b)$ and $B^{-1}(c)$ are, respectively, an initial segment and a final segment of $\mathbb{TP}^1$ with reference to the order $\preceq$ of $\mathbb{TP}^1$. This segments have an extremal point, b_0 and c_0 respectively. The restriction $B_{|[b_0,c_0]} : [b_0, c_0] \longrightarrow [b, c]$ is a projective isomorphism, hence $(c^1 - c^2) - (b^1 - b^2) = (c_0^1 - c_0^2) - (b_0^1 - b_0^2)$.

When we define the Hilbert metric we don't need to take the logarithms, as coordinates in tropical geometry already are in logarithmic scale. Hence the Hilbert metric on $\mathbb{TP}^1$ is simply the Euclidean metric:

$$d(x, y) = |(x^1 - x^2) - (y^1 - y^2)|$$

This definition can be extended to every separated tropical projective space $\mathbb{P}(M)$. If $a, b \in \mathbb{P}(M)$, we can choose two lifts $\bar{a}, \bar{b} \in M$. Then there is a unique linear map $\bar{f} : \mathbb{T}^2 \longrightarrow M$ such that $f(e_1) = \bar{b}, f(e_2) = \bar{a}$. The induced projective map $f : \mathbb{TP}^1 \longrightarrow \mathbb{P}(M)$ sends $0_{\mathbb{T}}$ in a and $\infty_{\mathbb{T}}$ in b. By corollary 120 the sets $f^{-1}(a)$ and $f^{-1}(b)$ are closed segments, with extremal points a_0, b_0. We can define the distance as $d(a, b) = d(a_0, b_0)$. It is easy to verify that this definition does not depend on the choice of the lifts $\bar{a}, \bar{b}$. Now we have to verify the triangular inequality, but it is more comfortable to give an example first.

For the projective spaces associated with the free modules we can calculate explicitly this distance. It is a well known distance, the Hilbert metric on the simplex in logarithmic coordinates.

Proposition 122. *Let $x, y \in \mathbb{TP}^{n-1}$. Then, for all lifts $\bar{x}, \bar{y} \in \mathbb{T}^n$:*

$$d(x, y) = \left(\bigoplus_{i=1}^{n} \bar{x}^i \oslash \bar{y}^i \right) \odot \left(\bigoplus_{i=1}^{n} \bar{y}^i \oslash \bar{x}^i \right) = \max_{i=1}^{n} (\bar{x}^i - \bar{y}^i) + \max_{i=1}^{n} (\bar{y}^i - \bar{x}^i)$$

Proof : The map $\bar{f}$ as above is defined in this case by the $2 \times n$ matrix:

$$\begin{pmatrix} y^1 & x^1 \\ \vdots & \vdots \\ y^n & x^n \end{pmatrix}$$

Then for all $h \in \mathbb{T}$,

$$\left(\bar{f}\left(\begin{smallmatrix} h \\ 1_{\mathbb{T}} \end{smallmatrix}\right)\right)^i = \max(y_i \odot h, x^i)$$

This is equal to x if $\forall i : h \leq x^i - y^i$, i.e. if $h \leq \min_i(x^i - y^i)$. As before, for all $k \in \mathbb{T}$,

$$\left(\bar{f}\left(\begin{smallmatrix} 1_{\mathbb{T}} \\ k \end{smallmatrix}\right)\right)^i = \max(y_i, x^i \odot k)$$

This is equal to y if $\forall i : k \leq y^i - x^i$, i.e. if $k \leq \min_i(y^i - x^i)$ Then, by definition

$$d(x, y) = |\min_i(x^i - y^i) + \min_i(y^i - x^i)|$$

By changing signs inside the absolute value, we have the thesis. □

From this explicit computation we can deduce easily that the triangular inequality holds for the distance we have defined in $\mathbb{TP}^{n-1}$, and that the topology induced by this distance on $\mathbb{TP}^{n-1}$ is the quotient of the product topology on $\mathbb{T}^n$.

Once we know that the triangular inequality holds for $\mathbb{TP}^{n-1}$, we can use this fact to prove it for all separated tropical projective spaces.

Proposition 123. *Let M be a separated $\mathbb{T}$-semimodule. Then the function $d : \mathbb{P}(M) \times \mathbb{P}(M) \longrightarrow \mathbb{R}_{\geq 0} \cup \{\infty\}$ satisfy:*

$$\forall x, y, z \in \mathbb{TP}^{n-1} : d(x, y) \leq d(x, z) + d(z, y)$$

Proof : Fix lifts $\bar{x}, \bar{y}, \bar{z} \in M$. We can construct a map $f : \mathbb{T}^3 \longrightarrow M$ such that $f(e_1) = x, f(e_2) = y, f(e_3) = z$. By lemma 121 there exist points $h_1, h_2, h_3 \in \mathbb{T}^3$ such that f is injective over $\mathrm{Span}_{\mathbb{T}}(h_i, h_j)$. Then $d(\pi(h_i), \pi(h_j)) = d(\pi(f_i), \pi(f_j))$. As the triangular inequality holds in $\mathbb{TP}^2$, then it holds for x, y, z. □

The metric we have defined for separated tropical projective spaces can achieve the value $+\infty$. Given a $\mathbb{T}$-semimodule M we can define the following equivalence relation on $M \setminus \{0\}$:

$$x \sim y \Leftrightarrow d(\pi(x), \pi(y)) < +\infty$$

The union of $\{0\}$ with one of these equivalence classes is again a $\mathbb{T}$-semimodule, and their projective quotients are tropical projective spaces with an ordinary (i.e. finite) metric.

For example in the free $\mathbb{T}$-semimodules $\mathbb{T}^n$ the equivalence class of the point $(1_{\mathbb{T}}, \ldots, 1_{\mathbb{T}})$ is the set $F\mathbb{T}^n$, and its associated projective space is $F\mathbb{TP}^{n-1}$, a tropical projective space in which the metric is finite.

For the $\mathbb{T}$-semimodule U^n an equivalence class is FU^n, and its associated projective space is FP^{n-1}, a tropical projective space in which the metric is finite.

We can calculate more explicitly the metric for FP^{n-1}. Let $x, y \in FP^{n-1}$ and let $\bar{x}, \bar{y} \in U^n$ be their lifts. By lemma 115 there exists a basis $\mathcal{E} = (e_1, \dots, e_n)$ of $\bar{x}$ such that $a_1 e_1, \dots a_n e_n$ is a basis of $\bar{y}$. In the tropical chart $\varphi_{\mathcal{E}}$, the point $\bar{x}$ has coordinates $(1_{\mathbb{T}}, \dots, 1_{\mathbb{T}})$, while the point $\bar{y}$ has coordinates $(\tau(a_1), \dots, \tau(a_n))$. Hence

$$d(x, y) = \max_i(\tau(a_i)) - \min_i(\tau(a_i))$$

6.5.3 Homotopy properties

In this section we will show that every separated tropical projective space with a finite metric is contractible.

If (X, d) is a metric space, we denote by $C^0([0,1], X)$ the space of continous curves in X, with the metric defined by

$$d(\gamma, \gamma') = \max_{t \in [0,1]} d(\gamma(t), \gamma'(t))$$

Note that the following pairing is continuous

$$C^0([0,1], X) \times [0,1] \ni (\gamma, t) \longrightarrow \gamma(t) \in X$$

Lemma 124. *Let (X, d) be a metric space and suppose we can construct a continuous map:*

$$C : X \times X \ni (x, y) \longrightarrow C_{x,y} \in C^0([0,1], X)$$

such that

1. *$C_{x,y}(0) = x$ and $C_{x,y}(1) = y$*
2. *$C_{x,x}$ is a constant curve.*

Then X is contractible.

Proof: We can construct a retraction $H : X \times [0,1] \longrightarrow X$ retracting X on one of its points $\{\bar{x}\}$ as

$$H(y, t) = C_{y,\bar{x}}(t)$$

By definition of C we have that $H(y, 0) = y$ and $H(y, 1) = \bar{x}$, and H is continuous as it is a composition of continuous functions. □

Lemma 125. *Let $x, y, a, b \in \mathbb{T}^n$ and let $\phi_{x,a}$ and $\phi_{y,b}$ be, respectively, the linear maps $\mathbb{T}^2 \longrightarrow \mathbb{T}^n$ defined by the matrices:*

$$\phi_{x,a} = \begin{pmatrix} x^1 & a^1 \\ \vdots & \vdots \\ x^n & a^n \end{pmatrix}, \qquad \phi_{y,b} = \begin{pmatrix} y^1 & b^1 \\ \vdots & \vdots \\ y^n & b^n \end{pmatrix}$$

Then

$$\forall v \in \mathbb{T}^2 : d(\pi(\phi_{x,a}(v)), \pi(\phi_{y,b}(v))) \leq \max(d(\pi(x), \pi(y)), d(\pi(a), \pi(b)))$$

Proof : Without loss of generality we can suppose that $v = (t, 1_{\mathbb{T}})$, so that $(\phi_{x,a}(v))^i = \max(x^i + t, a^i)$. Then

$$d(\pi(\phi_{x,a}(v)), \pi(\phi_{y,b}(v))) =$$

$$= \max_i(\max(x^i+t, a^i) - \max(y^i+t, b^i)) + \max_i(\max(y^i+t, b^i)) - \max(x^i+t, a^i)$$

It is easy to check that $\max(x^i+t, a^i) - \max(y^i+t, b^i)) \leq \max(x^i - y^i, a^i - b^i)$ by analyzing the four cases. □

Proposition 126. *For every separated $\mathbb{T}$-module M, its associated projective space $\mathbb{P}(M)$ is contractible with reference to the topology induced by the canonical metric.*

Proof : We have to construct a map C as in lemma 124. We will use tropical segments, rescaling their parametrization to the interval $[0, 1]$. If $x, y \in \mathbb{P}(M)$, we take lifts $\bar{x}, \bar{y} \in M$ and the map $\bar{f} : \mathbb{T}^2 \longrightarrow M$ s.t. $\bar{f}(e_1) = x, \bar{f}(e_2) = y$. As usual $f : \mathbb{TP}^1 \longrightarrow \mathbb{P}(M)$ is the induced map. By corollary 120 the sets $f^{-1}(x)$ and $f^{-1}(y)$ are closed segments, with extremal points x_0, y_0, hence f restricted to the interval $[x_0, y_0]$ is a curve joining x and y. Let ϕ be the affine map from the interval $[x_0, y_0]$ to the interval $[0, 1]$. We define $C_{x,y}$ as the reparametrization of f by ϕ. Properties 1 and 2 of the lemma 124 holds for C. To prove 3 we can show that:

$$\forall x, y, z, w \in \mathbb{P}(M) : \forall t \in [0, 1] : d(C_{x,y}(t), C_{z,w}(t)) \leq 3 \max(d(x, z), d(y, w))$$

To do this we take lifts $\bar{x}, \bar{y}, \bar{z}, \bar{w} \in M$, and a map $\bar{f} : \mathbb{T}^4 \longrightarrow M$ s.t. $f(e_1) = x, f(e_2) = y, f(e_3) = z, f(e_4) = w$. By lemma 121 there exist points $h_1, h_2, h_3, h_4 \in \mathbb{T}^4$ such that f is injective over $\mathrm{Span}_{\mathbb{T}}(h_i, h_j)$. Then $d(\pi(h_i), \pi(h_j)) = d(\pi(f_i), \pi(f_j))$. Moreover f is 1-Lipschitz on $\pi(\mathrm{Span}_{\mathbb{T}}(h_1, \ldots, h_4))$, hence our property on M follows from the same property on $\mathbb{T}^4$, and this follows from lemma 125. □

Chapter 7

Interpretation of the boundary points

In chapter 5 we have seen how it is possible to define a boundary for the Teichmüller spaces $\mathcal{T}^{cf}_{\mathbb{H}^2}(S)$ and for the spaces of convex projective structures $\mathcal{T}^{c}_{\mathbb{RP}^n}(M)$ using an inverse limit of logarithmic limit sets of these spaces, in other words we constructed the tropical counterparts of the spaces $\mathcal{T}^{cf}_{\mathbb{H}^2}(S)$ and $\mathcal{T}^{c}_{\mathbb{RP}^n}(M)$. The aim of this chapter is to give a geometric interpretation of the points of these tropical counterparts.

Every point of the boundary corresponds to a class of representations of the fundamental group of the manifold in $SL_n(\mathbb{F})$, where $\mathbb{F}$ is real closed non-archimedean field. Every such representation induces an action by tropical projective maps on our tropical projective spaces P^n. We compute the length spectrum of these actions on P^n, and we show that the length spectrum of this action identifies the boundary point. Then we use the fact that tropical projective spaces are contractible to show that for every action of the fundamental group of the manifold on a tropical projective space there exists an equivariant map from the universal covering of the manifold to the tropical projective space (see theorem 133).

This theorem can hopefully lead to interesting consequences about the interpretation of the boundary points. For example in the case $n = 1$, where P^1 is a real tree, the equivariant map induces a duality between actions of the fundamental group on P^1 and measured laminations on the surface. See the papers of Morgan and Shalen [MS84], [MS88] and [MS88'] for a reference about this fact.

It would be very interesting to extend this result in the general case. For example an action of the fundamental group of the surface on a tropical projective space P^n induces a degenerate metric on the surface, and this metric can be used to associate a length with each curve. Anyway it is not clear up to now how to classify these induced structures. This is closely related to a problem raised by J. Roberts (see [Oh01, problem 12.19]): how to extend the theory of measured laminations to higher rank groups, such as, for example, $SL_n(\mathbb{R})$.

7.1 Tropicalization of group representations

Let Γ be a group and $\rho : \Gamma \longrightarrow GL_{n+1}(\mathbb{F})$ be a representation of Γ in the general linear group of a non-archimedean field.

The group $GL_{n+1}(\mathbb{F})$ acts by linear maps on the tropical modules $U^{n+1}(\mathbb{F})$ and $FU^{n+1}(\mathbb{F})$, and by tropical projective maps on the tropical projective spaces $P^n(\mathbb{F})$ and $FP^n(\mathbb{F})$. The representation ρ defines an action of Γ on $FP^n(\mathbb{F})$.

For every matrix $A \in GL_{n+1}(\mathbb{F})$, we can define the **translation length** of A as:

$$l(A) = \inf_{x \in FP^n(\mathbb{F})} d(x, Ax)$$

Proposition 127. *Let $x \in FP^n$, and $L \subset V$ be a lift of x in FU^{n+1}. We denote by $e_1, \dots, e_{n+1}$ a basis of L, and by $\tilde{A}$ the matrix corresponding to A in this basis. Then*

$$d(x, A(x)) = \max_{i,j} \tau((\tilde{A})^i_j) + \max_{i,j} \tau((\tilde{A}^{-1})^i_j)$$

Proof: By lemma 115 applied to the $\mathcal{O}$-modules L and $A(L)$, there exist a basis $v_1, \dots, v_n$ of L and scalars $\lambda_1, \dots, \lambda_n \in \mathbb{F}$ such that $\lambda_1 v_1, \dots, \lambda_n v_n$ is a basis of $A(L)$. Then $d(x, Ax) = \max_i(\tau(\lambda_i)) - \min_i(\tau(\lambda_i))$. We will denote by M_1 the transition matrix from $e_1, \dots, e_n$ to $v_1, \dots, v_n$. As they are bases of the same $\mathcal{O}$-module, M_1 is in $GL_n(\mathcal{O})$. We will denote by M_2 the transition matrix from $\lambda_1 v_1, \dots, \lambda_n v_n$ to $A(e_1), \dots, A(e_n)$, and it is again in $GL_n(\mathcal{O})$. Let Δ be the diagonal matrix:

$$\Delta = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$$

Then the following relations hold:

$$\tilde{A} = M_2 \Delta M_1 \qquad \Delta = M_2^{-1} \tilde{A} M_1^{-1}$$

$$\tilde{A}^{-1} = M_1^{-1} \Delta^{-1} M_2^{-1} \qquad \Delta^{-1} = M_1 \tilde{A}^{-1} M_2$$

Hence:

$$\max_i(\tau(\lambda_i)) = \max_{i,j} \tau((\tilde{A})^i_j)$$

In the same way we have:

$$-\min_i(\tau(\lambda_i)) = \max_{i,j} \tau((\tilde{A}^{-1})^i_j)$$

□

The case $n = 1$ has been studied in [MS84].

Proposition 128. *Let $A \in SL_2(\mathbb{F})$. Then we have*

$$l(A) = 2\max(0, \tau(\mathrm{tr}(A)))$$

Proof : This is [MS84, prop. II.3.15]. The inequality $l(A) \geq 2\max(0, \tau(\mathrm{tr}(A)))$ follows from the previous proposition, as $\max_{i,j} \tau((\tilde{A})^i_j) \geq \tau(\mathrm{tr}(A))$ and $\mathrm{tr}(A) = \mathrm{tr}(A^{-1})$.

The reverse inequality follows by applying the previous proposition to the point x that is the class of a basis e_1, e_2 of $\mathbb{F}^2$ in which A appears in rational canonical form. Then

$$\tilde{A} = \begin{pmatrix} 0 & -1 \\ 1 & \mathrm{tr}(A) \end{pmatrix}$$

and by the previous proposition $d(x, A(x)) = 2\max(0, \tau(\mathrm{tr}(A)))$. □

Now, as in subsection 5.2.3, let $\mathbb{F}$ be a non-archimedean real closed field of finite rank extending $\mathbb{R}$, with a surjective real valuation $\overline{v} : \mathbb{F}^* \longrightarrow \mathbb{R}$ such that the valuation ring is convex. The field $\mathbb{K} = \mathbb{F}[i]$ is an algebraically closed field extending $\mathbb{C}$, with an extended valuation $\overline{v} : \mathbb{K}^* \longrightarrow \mathbb{R}$. We will use the notation $\tau = -\overline{v}$. We will also use the complex norm $|\cdot| : \mathbb{K} \longrightarrow \mathbb{F}_{\geq 0}$ defined by $|a + bi| = \sqrt{a^2 + b^2}$ and the conjugation $\overline{a + bi} = a - bi$.

If $A \in GL_{n+1}(\mathbb{K})$, we denote by $\lambda_1, \ldots, \lambda_{n+1}$ its eigenvalues, ordered such that $|\lambda_i| \geq |\lambda_{i+1}|$. We will denote $r(A) = |\lambda_1|$, the **spectral radius** of A.

Note that the function

$$|\cdot| : M_n(\mathbb{K}) \ni A \longrightarrow \in \max_{i,j} |A^i_j| \in \mathbb{F}_{\geq 0}$$

is a consistent norm on $M_n(\mathbb{K})$, hence, by the spectral radius theorem, we have $r(A) \leq |A|$.

Proposition 129. *Suppose the field $\mathbb{K}$ is as above. Then a matrix $A \in GL_{n+1}(\mathbb{K})$ acts on $FP^n(\mathbb{K})$. Then the* inf *in the definition of $l(A)$ is a minimum, and it is equal to*

$$l(A) = \tau\left(\left|\frac{\lambda_1}{\lambda_{n+1}}\right|\right)$$

Proof : By proposition 127 we have that for every $x \in FP^n(\mathbb{K})$

$$d(x, A(x)) \geq \tau(r(A)) + \tau(r(A^{-1}))$$

Hence

$$l(A) \geq \tau(r(A)) + \tau(r(A^{-1}))$$

or, in other words,
$$l(A) \geq \tau\left(\left|\frac{\lambda_1}{\lambda_{n+1}}\right|\right)$$

We only need to show that the lower bound of previous corollary is actually achieved. The Jordan form of A is

$$\begin{pmatrix} \lambda_1 & * & & \\ & \lambda_2 & \ddots & \\ & & \ddots & * \\ & & & \lambda_{n+1} \end{pmatrix}$$

where the entries marked by $*$ are 0 or 1. Let $v_1, \ldots, v_{n+1}$ be a Jordan basis, and let $L = \mathrm{Span}_{\mathcal{O}}(v_1, \ldots, v_n) \in U^{n+1}$. By proposition 127

$$d(\pi(L), A\pi(L)) = \tau\left(\left|\frac{\lambda_1}{\lambda_{n+1}}\right|\right)$$

□

Now suppose that $A \in GL_n(\mathbb{F})$, with $\mathbb{F}$ a non-archimedean real closed field as above. Hence A acts on $FP^n(\mathbb{F})$, and now we want to study the translation lenght of A over $FP^n(\mathbb{F})$. As before, we denote by $\lambda_1, \ldots, \lambda_{n+1} \in \mathbb{K}$ its eigenvalues, ordered such that $|\lambda_i| \geq |\lambda_{i+1}|$.

Proposition 130. *Suppose that $\mathbb{F}$ is as above, and that $A \in GL_{n+1}(\mathbb{F})$. We consider the translation lenght $l(A)$ with respect to the action of A on $FP^n(\mathbb{F})$. Then the* inf *in the definition of $l(A)$ is a minimum, and it is equal to*
$$l(A) = \tau\left(\left|\frac{\lambda_1}{\lambda_{n+1}}\right|\right)$$

Proof: As $FP^n(\mathbb{F}) \subset FP^n(\mathbb{K})$, by proposition 129 we have the inequality

$$l(A) \geq \tau\left(\left|\frac{\lambda_1}{\lambda_{n+1}}\right|\right)$$

To prove that this lower bound is achieved, we will choose a suitable basis, as above. Consider the decomposition into sum of generalized eigenspaces

$$\mathbb{K}^{n+1} = \sum_{i=1}^{n} \ker((A - \lambda_i \mathrm{Id})^{n+1})$$

For every $\lambda_i \in \mathbb{F}$, the generalized eigenspace $\ker((A - \lambda_i \mathrm{Id})^{n+1})$ has a basis of generalized eigenvectors in $\mathbb{F}^{n+1}$. If $\lambda_i \in \mathbb{K} \setminus \mathbb{F}$, then $\overline{\lambda_i}$ is an eigenvalue, and if $v_1, \ldots, v_s$ is a basis of generalized eigenvectors of $\ker((A - \lambda_i \mathrm{Id})^{n+1})$, then $\overline{v_1}, \ldots, \overline{v_s}$ is a basis of generalized eigenvectors of $\ker((A - \overline{\lambda_i} \mathrm{Id})^{n+1})$.

The vectors $v_i + \overline{v_i}$ and $v_i - \overline{v_i}$ are in $\mathbb{F}^{n+1}$, and they form a basis of $\ker((A-\lambda_i \mathrm{Id})^{n+1}) + \ker((A-\overline{\lambda_i}\mathrm{Id})^{n+1})$. In this way we have constructed a basis $v_1, \dots, v_{n+1}$ of $\mathbb{F}^{n+1}$ such that $|A| = |\lambda_1|$ and $|A^{-1}| = |\lambda_{n+1}|$. Let $L = \mathrm{Span}_{\mathcal{O}}(v_1, \dots, v_n) \in U^{n+1}$, then, by proposition 127

$$d(\pi(L), A\pi(L)) = \tau\left(\left|\frac{\lambda_1}{\lambda_{n+1}}\right|\right)$$

□

7.2 Boundary points

Let $\overline{S} = \Sigma_g^k$, a surface of genus g with $k \geq 0$ boundary components and such that $\chi(\overline{S}) < 0$, and let S be the interior part of $\overline{S}$. In subsection 5.3.1 we considered the family $\mathcal{G} = \{J_\gamma\}_{\gamma \in \pi_1(S)}$, and we constructed a compactification of $\mathcal{T}^{cf}_{\mathbb{H}^2}(S)$:

$$\overline{\mathcal{T}^{cf}_{\mathbb{H}^2}(S)}_{\mathcal{G}} = \mathcal{T}^{cf}_{\mathbb{H}^2}(S) \cup \partial_{\mathcal{G}} \mathcal{T}^{cf}_{\mathbb{H}^2}(S)$$

The cone over the boundary $C(\partial_{\mathcal{G}} \mathcal{T}^{cf}_{\mathbb{H}^2}(S))$ can be identified with a subset of $\mathbb{R}^{\mathcal{G}} = \mathbb{R}^{\pi_1(S)}$.

Every action of $\pi_1(S)$ on a tropical projective space $FP^1(\mathbb{F})$ has a well defined length spectrum $(l(\gamma))_{\gamma \in \pi_1(S)} \in \mathbb{R}^{\pi_1(S)}$.

Theorem 131. *Let $\mathbb{F} = H(\overline{\mathbb{R}}^{\mathbb{R}})$, the Hardy field as in subsection 4.1.1. The points of $C(\partial_{\mathcal{G}} \mathcal{T}^{cf}_{\mathbb{H}^2}(S))$ are length spectra of actions of the fundamental group $\pi_1(S)$ on the tropical projective space $FP^1(\mathbb{F})$.*

Proof : The semi-algebraic set $\mathcal{T}^{cf}_{\mathbb{H}^2}(M)$ has an extension to the field $\mathbb{F}$, that we will denote by $\overline{\mathcal{T}^{cf}_{\mathbb{H}^2}(S)} \subset \overline{\mathrm{Char}}(\pi_1(S), SL_2(\mathbb{F}))$. Every element of $\overline{\mathcal{T}^{cf}_{\mathbb{H}^2}(S)}$ is a conjugacy class of a representation $\rho : \pi_1(S) \longrightarrow SL_2(\mathbb{F})$. The representation ρ has all the properties of the representations in $\mathcal{T}^{cf}_{\mathbb{H}^2}(S)$ that can be expressed by a first order formula, for example, if $\gamma \in \pi_1(S)$, the matrix $\rho(\gamma)$ has $|\operatorname{tr}(\rho(\gamma))| \geq 2$.

Let $x \in C(\partial_{\mathcal{G}} \mathcal{T}^{cf}_{\mathbb{H}^2}(S))$. By proposition 90, there exists a representation $\rho \in \overline{\mathcal{T}^{cf}_{\mathbb{H}^2}(S)}$ such that for every $\gamma \in \pi_1(S)$, the matrix $\rho(\gamma) \in SL_2(\mathbb{F})$ satisfies $x_{J_\gamma} = \tau(\operatorname{tr}(\rho(\gamma)))$.

Consider the action of $\pi_1(S)$ on $FP^n(\mathbb{F})$ induced by the representation ρ. By proposition 128, the translation length of an element γ with respect of this action is $l(\rho(\gamma)) = 2\max(0, \tau(\operatorname{tr}(A)))$. As $|\operatorname{tr}(\rho(\gamma))| \geq 2$, we have

$$l(\rho(\gamma)) = 2\tau(\operatorname{tr}(A))$$

□

In the same way, we give a geometric interpretation to the points of the boundaries of the spaces of convex projective structures. Let M be a closed n-manifold such that the fundamental group $\pi_1(M)$ has trivial virtual center, it is Gromov hyperbolic, and it is torsion free (note that every closed hyperbolic n-manifold whose fundamental group is torsion-free satisfies the hypotheses). In subsection 5.3.2 we considered the family $\mathcal{G} = \{e_\gamma\}_{\gamma\in\pi_1(M)}$, and we constructed a compactification of $\mathcal{T}^c_{\mathbb{RP}^n}(M)$:

$$\overline{\mathcal{T}^c_{\mathbb{RP}^n}(M)}_{\mathcal{G}} = \mathcal{T}^c_{\mathbb{RP}^n}(M) \cup \partial_{\mathcal{G}}\mathcal{T}^c_{\mathbb{RP}^n}(M)$$

The cone over the boundary $C(\partial_{\mathcal{G}}\mathcal{T}^c_{\mathbb{RP}^n}(M))$ can be identified with a subset of $\mathbb{R}^{\mathcal{G}} = \mathbb{R}^{\pi_1(M)}$.

Every action of $\pi_1(M)$ on a tropical projective space $FP^n(\mathbb{F})$ has a well defined length spectrum $(l(\gamma))_{\gamma\in\pi_1(M)} \in \mathbb{R}^{\pi_1(M)}$.

Theorem 132. *Let $\mathbb{F} = \mathbb{R}((t^{\mathbb{R}^r}))$, where r is the dimension of $\mathcal{T}^c_{\mathbb{RP}^n}(M)$ (see the definition in subsection 5.2.3). The points of $C(\partial_{\mathcal{G}}\mathcal{T}^c_{\mathbb{RP}^n}(M))$ are length spectra of actions of the fundamental group $\pi_1(M)$ on the tropical projective space $FP^n(\mathbb{F})$.*

Proof : The semi-algebraic set $\mathcal{T}^c_{\mathbb{RP}^n}(M)$ has an extension to the field $\mathbb{F}$, that we will denote by $\overline{\mathcal{T}^c_{\mathbb{RP}^n}(M)} \subset \overline{\mathrm{Char}}(\pi_1(M), SL_{n+1}(\mathbb{F}))$. Every element of $\overline{\mathcal{T}^c_{\mathbb{RP}^n}(M)}$ is a conjugacy class of a representation $\rho : \pi_1(M) \longrightarrow SL_{n+1}(\mathbb{F})$.

Let $x \in C(\partial_{\mathcal{G}}\mathcal{T}^c_{\mathbb{RP}^n}(M)) \subset \mathbb{R}^{\mathcal{G}}$. As we said in subsection 5.2.3, there exists a representation $\rho \in \overline{\mathcal{T}^c_{\mathbb{RP}^n}(M)}$ such that for every $\gamma \in \pi_1(M)$, the matrix $\rho(\gamma)$ satisfies $x_{e_\gamma} = \tau\left(\left|\frac{\lambda_1}{\lambda_{n+1}}\right|\right)$.

Consider the action of $\pi_1(M)$ on $FP^n(\mathbb{F})$ induced by the representation ρ. By proposition 130, the translation length of an element γ with respect of this action is $l(\rho(\gamma)) = \tau\left(\left|\frac{\lambda_1}{\lambda_{n+1}}\right|\right)$. □

7.3 The equivariant map

These actions of $\pi_1(M)$ on the tropical projective spaces FP^n should correspond to some kind of dual structures on the surface.

Suppose that Γ is the fundamental group of an n-manifold M, and suppose that $\pi_2(M) = \cdots = \pi_{n-1}(M) = 0$. Note that if M is a surface, this hypothesis is empty. For example every n-manifold M whose universal covering is $\mathbb{R}^n$ satisfy this hypothesis, in particular every manifold admitting a convex projective structure. We will denote by $p : \tilde{M} \longrightarrow M$ the universal covering of M. Then suppose that Z is a simply connected topological space with an action of Γ. It is always possible to construct an equivariant map:

Theorem 133. *There exists a map $f : \tilde{M} \longrightarrow Z$ that is equivariant for the action of Γ, i.e.*

$$\forall x \in \tilde{M} : \forall\gamma \in \Gamma : \gamma(f(x)) = f(\gamma(x))$$

Proof : The group Γ acts diagonally on the space $\tilde{M} \times Z$:

$$\gamma(x, z) = (\gamma(x), \gamma(z))$$

This action is free and proper, $\tilde{M} \times Z$ is simply connected, hence

$$P : \tilde{M} \times Z \longrightarrow \mathcal{K} = (\tilde{M} \times Z)/\Gamma$$

is a universal cover, and $\pi_1(\mathcal{K}) = \Gamma$.

As M is a manifold it is homeomorphic to a CW-complex of dimension n with only one 0-cell. Hence the hypothesis that $\pi_2(M) = \cdots = \pi_{n-1}(M) = 0$ implies that the isomorphism $\pi_1(M) \longrightarrow \pi_1(\mathcal{K})$ is induced by a map $\psi : M \longrightarrow \mathcal{K}$, well defined up to homotopy.

As $\tilde{M}$ is simply connected, we can lift the map $\phi = \psi \circ p : \tilde{M} \longrightarrow \mathcal{K}$ to a map $\tilde{\phi} : \tilde{M} \longrightarrow \tilde{M} \times Z$ such that $P \circ \tilde{\phi} = \phi$. The equivariant map f we are searching for is the composition of $\tilde{\phi}$ with the projection on Z. We have to check that it is equivariant, and to show this we will prove that $\tilde{\phi}$ is equivariant. We need to prove that:

$$\forall y \in \tilde{M} : \forall \gamma \in \Gamma : \gamma(\tilde{\phi}(y)) = \tilde{\phi}(\gamma(y))$$

Fix an $y \in \tilde{M}$ and a $\gamma \in \Gamma$. Let $x_0 = p(y) = p(\gamma(y)) \in M$ and let $x_1 = \psi(x_0) = P(\tilde{\phi}(y)) = P(\tilde{\phi}(\gamma(y)))$ (as $\tilde{\phi}$ is a lift of $\psi : M \longrightarrow \mathcal{K}$).

Now we identify Γ with the based fundamental groups $\pi_1(M, x_0)$ and $\pi_1(\mathcal{K}, x_1)$. By the definition of ψ, with this identification, the isomorphism $\psi_* : \pi_1(M, x_0) \longrightarrow \pi_1(\mathcal{K}, x_1)$ is the identity, hence $\psi_*(\gamma) = \gamma$.

Consider the lift $\tilde{\gamma}$ of the path γ in $\tilde{M}$ starting from the point y. The other extreme of $\tilde{\gamma}$ is the point $\gamma(y)$. The same way the lift $\widetilde{\psi_*(\gamma)}$ of the path $\psi_*(\gamma)$ in $\tilde{M} \times \mathcal{K}$ starting from the point $\tilde{\phi}(y)$ is the image $\tilde{\phi}(\tilde{\gamma})$, hence the other extreme of this path is the point $\tilde{\phi}(\gamma(y))$. This is precisely the definition of $\gamma(\tilde{\phi}(y))$. $\square$

Suppose that M is an n-manifold as above, and that we have an action $\pi_1(M) \times P^m \longrightarrow P^m$. Let $\widetilde{M} \longrightarrow M$ be the universal covering of M. As P^m is a contractible space there is a $\pi_1(M)$-equivariant map

$$f : \widetilde{M} \longrightarrow P^n$$

An interesting open problem is to understand the dual structure this equivariant map induces on M.

The case where M is an hyperbolic surface and $m = 1$ has been studied by Morgan and Shalen in [MS88] and it is well understood: P^1 is a real tree and the equivariant map induces a measured lamination on M, that is dual to the action.

This work can possibly lead to the discovery of analogous structures for the general case. For example an action of the fundamental group of the manifold on a tropical projective space P^m induces a degenerate metric on the surface, and this metric can be used to associate a length with each curve. Anyway it is not clear up to now how to classify these induced structures. This is closely related to a problem raised by J. Roberts (see [Oh01, problem 12.19]): how to extend the theory of measured laminations to higher rank groups, such as, for example, $SL_n(\mathbb{R})$.

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